Normal Dilations

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Abstract

Normal dilations usually serve as a sort of standard models for the intrinsic structure of Hilbert space operators. Here, we will look into some old and new results of normal dilations related to numerical ranges and spectral sets.

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1 Introduction

Let $\mathcal{H}$ be a Hilbert space equipped with the inner product (x, y), and let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators acting on $\mathcal{H}$ equipped with the operator norm

$$\|A\| = \sup\{\|Ax\| : x \in \mathbf{C}^n,\ (x,x) = 1\}.$$

If $\mathcal{H}$ is n-dimensional, we identify $\mathcal{H}$ with $\mathbf{C}^n$ and $\mathcal{B}(\mathcal{H})$ with the algebra M_n of $n \times n$ complex matrices. The *numerical range* of an operator $A \in \mathcal{B}(\mathcal{H})$ is defined by

$$W(A) = \{(Ax, x) : x \in \mathcal{H},\ (x,x) = 1\}.$$

The spectrum of A is denoted by $\sigma(A)$.

We say that $A \in \mathcal{B}(\mathcal{H})$ has a *dilation* $\tilde{A} \in \mathcal{B}(\tilde{\mathcal{H}})$ if $A = V^*\tilde{A}V$ for some isometry $V : \mathcal{H} \to \tilde{\mathcal{H}}$; equivalently, $\tilde{A}$ is unitarily similar to a 2×2 operator-matrix of the form $\begin{pmatrix} A & * \\ * & * \end{pmatrix}$. In such a case, A is called a *compression* of $\tilde{A}$.

Notably, normal dilations arises in structure theory as a sort of *non-commutative spectral decomposition* in terms of *a non-commutative resolution of the identity* . Namely, if A has a normal dilation $\tilde{A}$ with spectral projections $E(\cdot)$, and if $Q(\mathcal{S})$ is defined as the compression of $E(\mathcal{S})$ on the Hilbert space $\mathcal{H}$ for each Borel subset $\mathcal{S}$ of the the spectrum $\sigma(\tilde{A})$, then we get

$$I = \int dQ \quad \text{and} \quad A = \int \lambda dQ.$$

In case that the normal dilation has a finite spectrum $\lambda_1, ...\lambda_n$, we can write the non-commutative resolution of identity in terms of finitely many positive operators Q_j, such that

$$I_n = \sum_{j=1}^{n} Q_j \quad \text{and} \quad A = \sum_{j=1}^{n} \lambda_j Q_j.$$

This lecture note is organized as follows. In Section 2, we examine normal dilations of finite spectra, with a detailed study of the underpinnings of the Mirman Theorem (Theorem 2.1). In Section 3, we look into the structure of unitary dilations with particular concerns about constrained unitary dilations . Section 4 includes a description of recent results of the structure theory of joint spectral circles in connection with simultaneous normal dilations for a pair of operators.

2 Normal dilations with finite spectra

Let K be a convex compact subset of $\mathbf{C}$. It has been a major structure problem to determine whether any operator A can have a normal dilation $\tilde{A}$ whose spectrum is a subset of K. Obviously, we have numerical range inclusion $W(A) \subseteq W(\tilde{A}) \subseteq K$. It is natural to ask whether the numerical inclusion $W(A) \subseteq K$ suffices to infer that A has a normal dilation with spectrum as a subset of K. It turns out the answer is true only for the case K is a triangle (including the degenerate case when K is a line segment or a point).

If K is a line segment or a single point, then the condition $W(A) \subseteq K$ implies that A is a normal operator whose spectrum is a subset of K. Next theorem of Mirman ([11], see also [12], [3])is a case of great significance in structure theory.

Theorem 2.1 (Mirman). *Let $A \in \mathcal{B}(\mathcal{H})$ and let $\gamma_1, \gamma_2, \gamma_3 \in \mathbf{C}$. The following conditions are equivalent.*

(a) *$W(A)$ is included in the triangle with vertices $\gamma_1, \gamma_2, \gamma_3$.*

(b) *$A = V^*(B \otimes I)V$, where $B = \operatorname{diag}(\gamma_1, \gamma_2, \gamma_3)$, I is the identity operator on the Hilbert space $\mathcal{H}$, and $V : \mathcal{H} \to \mathbf{C}^3 \otimes \mathcal{H}$ is an operator satisfying $V^*V = I$.*

Idea of Proof. The quickest proof of (a) $\Rightarrow$ (b) is to make use of an affine real transform as shown in [3, Proof of Prop. 2.3]. Note that if $\Phi(x+iy) = (ax+by+r)+i(cx+dy+s)$ with $a, b, c, d, r, s \in \mathbf{R}$ is an affine transform on $\mathbf{C}$, then for each $A = A_1 + iA_2 \in \mathcal{B}(\mathcal{H})$ such that A_1 and A_2 are self-adjoint operators, we can define

$$\Phi(A) = (aA_1 + bA_2 + rI) + i(cA_1 + dA_2 + sI).$$

Now, if the given three points γ_j are not collinear, we can choose an affine transform so that $\Phi(\gamma_1) = 0, \Phi(\gamma_2) = 1$, and $\Phi(\gamma_3) = i$. Hence, by the condition (a), it follows that $\Phi(A) = H + iG$ with $H \geq O, G \geq O$ and $I \geq H + G$. Set $V = (\sqrt{I-H-G} \quad \sqrt{H} \quad \sqrt{G})^*$. Then $V^*V = I$ and $A = V^*(B \otimes I)V$. □

The following rather simple result is related to some interesting geometrical features.

Proposition. 2.2. *Let A be a 2×2 matrix. Then A can be dilated to a 3×3 unitary matrices $\tilde{A}$ iff the operator norm of A is exactly one.*

Proof. Suppose A is of norm 1. By unitary similarity, we may assume that A attains its norm at the first column and write $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. By the orthogonal process, we can fill in appropriate entries so as to get a 3×3 unitary matrix $\tilde{A} = \begin{pmatrix} a_{11} & a_{12} & * \\ a_{21} & a_{22} & * \\ 0 & * & * \end{pmatrix}$.

Conversely, suppose that U is a 3×3 unitary matrix. Let $\mathcal{S}$ be any two-dimensional subspace of the underlying 3-dimensional Hilbert space. Then $dim(U(\mathcal{S}) \cap \mathcal{S}) \geq 1$; i.e.,there exists a nonzero-vector $v \in \mathcal{S}$ such that $Uv \in \mathcal{S}$, so the compression of U to the space $\mathcal{S}$ is a 2×2 matrix of norm one as desired. □

Remark 2.3. It is well known that the numerical ranges determine all 2×2 matrices up to unitary similarity. If the given 2×2 matrix A is normal , then $W(A)$ is the line segment (joining two eigenvalues) or the singleton set (of repeated eigenvalues). If A is non-normal, then A is unitarily similar to a 2×2 upper-triangular matrix $\begin{pmatrix} \mu_1 & c \\ 0 & \mu_2 \end{pmatrix}$ with a unique positive number c; thus $W(A)$ is the closed elliptical disk with two eigenvalues as foci and the value c as the length of the minor axis. More fascinating geometrical features associated with Prop. 2.2 are described as follows:

(a) For each pair of complex numbers μ_1 and μ_2 of modulus smaller than 1, there exists a unique positive real number c so that the operator norm of $\begin{pmatrix} \mu_1 & c \\ 0 & \mu_2 \end{pmatrix}$ is exactly one; in fact, c satisfies the equality $c^2 = (1 - |\mu_1|^2)(1 - |\mu_2|^2)$. This is the same as to say that, among all 2×2 matrices with μ_1 and μ_2 as two fixed eigenvalues, there is only one matrix A (up to unitary similarity) of norm 1. Since A can be dilated to a 3×3 unitary matrix U, it follows that $W(A) \subseteq \Delta$ where Δ is a triangle with three vertices on the unit circle. With respect to infinitely many different 3×3 unitary dilations (as we can multiply the third column of each 3×3 unitary matrix by any complex number of modulus 1), there are infinitely many different triangles $\Delta \supseteq W(A)$ such that the three vertices of each Δ lie on the unit circle Γ.

(b) In other words, among all elliptical disks with common foci μ_1 and μ_2, there is only one ellipse $\mathcal{E}$ of which the length of the minor axis is $\sqrt{(1 - |\mu_1|^2)(1 - |\mu_2|^2)}$. Such an elliptical disk enjoys a subtle geometrical property: there is a triangle Δ inscribing the unit circle Γ and circumscribing the elliptical disk $\mathcal{E}$.

(c) On the other hand, suppose there is an elliptical disk $\mathcal{E}$ circumscribed by a triangle Δ with the three vertices on the unit circle Γ. Then the minor axis of $\mathcal{E}$ must be of the length $\sqrt{(1 - |\mu_1|^2)(1 - |\mu_2|^2)}$ where μ_1 and μ_2 are foci of $\mathcal{E}$. Moreover, given each $\gamma_1 \in \Gamma$, there exist two unique points γ_2 and γ_3 (depending on γ_1) $\in \Gamma$ so that $\mathcal{E}$ is circumscribed by the triangle formed by $\gamma_1, \gamma_2, \gamma_3$. Since γ_1 is an arbitrary point on the unit circle, there are infinitely many triangles Δ with vertices on the unit circle so that each Δ circumscribes the given ellipse $\mathcal{E}$.

(d) For each fixed triangle Δ with three vertices on the unit circle, there are infinitely many elliptical disks circumscribed by Δ. This is equivalent to say that each 3×3 unitary matrix has infinitely many compressions on 2-dimensional subspaces, and furthermore, each of such compression is a 2×2 matrix of norm 1.

The Mirman Theorem is related to a numerical range in the shape of a triangle. The analogous statement for a square is invalid as shown in the following:

Example 2.4 ([3]. Let $A = \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}$ and let $B = \mathrm{diag}\,(1, -1, i, -i)$. Then $W(A) \subseteq W(B)$, where $W(A)$ is the circle centered at the origin with radius $\sqrt{2}/2$ and $W(B)$ is the square with four vertices $1, -1, i, -i$. However, A cannot be dilated to an operator of the form $B \otimes I$ as $\|A\| > \|B\|$.

In connection to the above example, we have the next theorem (as a simplification of [3, Theorem 2.5]) characterizing operators admitting dilations to normal operators whose spectra are subsets of $\{1, -1, i, -i\}$.

Theorem 2.5. *Let $A \in \mathcal{B}(\mathcal{H})$. The following conditions are equivalent.*

(a) *A has a dilation of the form $B = \mathrm{diag}\,(1, -1, i, -i) \otimes I$.*

(b) *There exist four positive operators Q_1, Q_2, Q_3, Q_4 such that $Q_1 + Q_2 + Q_3 + Q_4 = I$ and $A = (Q_1 - Q_2) + i(Q_3 - Q_4)$.*

(c) *$\|A_1 + e^{i\theta} A_2\| \leq 1$ for all real θ, where $A = A_1 + iA_2$ with self-adjoint A_1 and A_2.*

Note that the condition (b) can be changed to a weaker condition

(b') *There exist four positive operators* Q'_1, Q'_2, Q'_3, Q'_4 *such that* $Q'_1 + Q'_2 + Q'_3 + Q'_4 \le I$ *and* $A = (Q'_1 - Q'_2) + i(Q'_3 - Q'_4)$.
In fact, the substitutions of $Q_j = Q'_j + (I - Q'_1 - Q'_2 - Q'_3 - Q'_4)/4$ will change the condition (b') back to the condition (b).

Also note that the condition (b) cannot be replaced by a stronger condition as

(**) *With respect to the canonical decomposition* $A = (A_1 - A_2) + i(A_3 - A_4)$ *where* A_1, A_2, A_3 *and* A_4 *are positive operator with* $A_1A_2 = O = A_3A_4$, *we have* $A_1 + A_2 + A_3 + A_4 \le I$.

Example 2.6. Consider the 2×2 matrix $A = \begin{pmatrix} 0.4 + 0.3i & 0.5 \\ 0.5 & 0.4 - 0.3i \end{pmatrix}$. Then with respect to the canonical decomposition, $A = \begin{pmatrix} 0.45 & 0.45 \\ 0.45 & 0.45 \end{pmatrix} - \begin{pmatrix} 0.05 & -0.05 \\ -0.05 & 0.05 \end{pmatrix} + i \begin{pmatrix} 0.3 & 0 \\ 0 & 0 \end{pmatrix} - i \begin{pmatrix} 0 & 0 \\ 0 & 0.3 \end{pmatrix}$ while the sum of the four positive semi-definite matrices is $\begin{pmatrix} 0.8 & 0.4 \\ 0.4 & 0.8 \end{pmatrix} \not\le I$; hence the condition (**) is not valid.

On the other hand , we can rewrite $A = \begin{pmatrix} 0.45 & 0.45 \\ 0.45 & 0.45 \end{pmatrix} - \begin{pmatrix} 0.05 & -0.05 \\ -0.05 & 0.05 \end{pmatrix} + i \begin{pmatrix} 0.4 & -0.2 \\ -0.2 & 0.1 \end{pmatrix} - i \begin{pmatrix} 0.1 & -0.2 \\ -0.2 & 0.4 \end{pmatrix}$ whence the sum of the four positive semi-definite matrices is exactly I; hence the condition (b) is valid.

Remark 2.7. By a linear transform, the square of four vertices $1, i, -1, -i$ can be changed to the square with four vertices $0, 1, 1 + i, i$. Then the analogous problem is to determine all pairs of positive operators A_1 and A_2 that admit commuting projections as simultaneous dilations. This is equivalent to the problem to determine the existence of a single operator Q satisfying four inequalities:

$$O \le Q, \quad A_1 + A_2 - I \le Q, \quad Q \le A_1, \quad Q \le A_2.$$

The necessary and sufficient condition (as an analogue to the statement of Theorem 2.5 (C)) turns out to be

$$\|A_1 + A_2 - I + e^{i\theta}(A_1 - A_2)\| \le 1$$

for all real θ.

Apparently, there is no easy structure theorem for general normal dilations of finite spectra. The following is sort of converse of the Mirman Theorem showing the best result along these lines.

Proposition 2.8. *Let* K *be a compact convex subset of* $\mathbf{C}$ *other than a triangle (or in the degenerate case as a line segment or a point). Then there exists a* 2×2 *matrix* A *with its numerical range* $W(A) \subseteq K$, *but* A *cannot be dilated to to normal operator* $\tilde{A}$ *with* $\sigma(\tilde{A}) \subseteq K$.

Idea of Proof. Let Γ be the circle (not necessarily to be centered at the origin) of smallest radius to surround the the compact convex set K. Then we can find an elliptical disk $\mathcal{E}$ included in K but there is no triangle Δ with three vertices on the circle Γ and Δ circumscribes $\mathcal{E}$. Specifically, suppose Γ is the circle centered at μ_0 of radius r; then $\mathcal{E}$ corresponds to a 2×2 matrix A such that $\|A - \mu_0 I\| > r$. Hence A cannot be dilated to any normal operator with its spectrum on the circle Γ. □

3 Unitary dilations

In [7], Halmos showed explicitly that each contraction $A \in \mathcal{B}(\mathcal{H})$ has a unitary dilation $U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ of the form

$$U = \begin{pmatrix} A & \sqrt{1 - AA^*} \\ \sqrt{1 - A^*A} & -A^* \end{pmatrix}.$$

This result has generated a lot of research, including the far reaching Sz.-Nagy dilation theorem [14]: Each contraction $A \in \mathcal{B}(\mathcal{H})$ has a power unitary dilation; i.e., there is a unitary U satisfying

$$U^k = \begin{pmatrix} A^k & * \\ * & * \end{pmatrix}, \qquad k = 1, 2, \ldots.$$

Notably, the fact "each contraction $A \in M_n$ has a unitary dilation $U \in M_{2n}$" can be deduced directly from the following simple and yet stronger result (see [4, Prop. 4.1]).

Proposition 3.1. *Let $A \in M_n$ be a contraction. Then $A = (U_1 + U_2)/2$ for some unitary matrices $U_1, U_2 \in M_n$; consequently, $A = V^*(U_1 \oplus U_2)V$ where $V = \frac{1}{\sqrt{2}}\begin{pmatrix} I_n \\ I_n \end{pmatrix}$.*

In this section, we are particularly concerned about the structure of a contraction $A \in \mathcal{B}(\mathcal{H})$ subject to a constraint $A + A^* \leq aI$ for some real a.

Theorem 3.2 (Choi and Li [4]). *Let $A \in \mathcal{B}(\mathcal{H})$ be a contraction such that $A + A^* \leq aI$ for some real number a. Then A has a unitary dilation $U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ satisfying $U + U^* \leq aI$. In the case when $\mathcal{H}$ is of dimension n, the matrix $U \in M_{2n}$ can be chosen such that its $2n$ eigenvalues are $e^{\pm i\theta_1}, \ldots, e^{\pm i\theta_n}$ with $2\cos\theta_j \leq a$ for all j (i.e., non-real eigenvalues occur in conjugate pairs, real eigenvalues have even multiplicities).*

Obviously, the case $A + A^* \leq aI$ with $a \geq 2$ is automatic while the case $A + A^* \leq aI$ with $a < -2$ is vacuous.

Remark 3.3. The following are related facts and immediate consequences of the constrained unitary dilations.

First, in the finite-dimensional case, each unitary dilation $U \in M_{2n}$ of a contraction $A \in M_n$ must be of the form $U = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ with $B = W_1\sqrt{I - A^*A}, C = -\sqrt{I - AA^*}W_2$, and $D = W_1 A^* W_2$ for some unitary $W_1, W_2 \in M_n$. Thus U is unitarily similar to

$$(I \oplus W_1^*)U(I \oplus W_1) = \begin{pmatrix} A & -\sqrt{I - AA^*}W_0 \\ \sqrt{I - A^*A} & A^*W_0 \end{pmatrix}$$

with $W_0 = W_2W_1$. In order to get a "constrained" unitary dilation, we need a judicious choice of unitary $W_0 \in M_n$. If A is normal, then we just choose $W_0 = I$. If A is non-normal, there is an algorithm (an account of non-commutative matrix manipulation) to construct W_0.

Secondly, if $A \in M_n$ is a real matrix, our construction will also yield a real constrained orthogonal dilation $U \in M_{2n}$.

Moreover, we can use Theorem 3.2 to get a spectral decomposition for non-normal constrained contractions in terms of "a non-commutative resolution of the identity". Here we state only the finite-dimensional case:

Corollary 3.4 (Choi and Li [4]). *Suppose $A \in M_n$ is a contraction satisfying $A + A^* \le aI_n$ for some real number a. Then there are n real numbers $\theta_1, \ldots, \theta_n \in [0, \pi]$ with $2\cos\theta_j \le a$ for all j, and positive semidefinite rank-1 matrices $Q_1, \ldots, Q_{2n} \in M_n$, such that*

$$I_n = \sum_{j=1}^{n} (Q_j + Q_{n+j}) \quad \text{and} \quad A = \sum_{j=1}^{n} \left(e^{i\theta_j} Q_j + e^{-i\theta_j} Q_{n+j} \right).$$

The constrained unitary dilation is particularly useful in the study of numerical ranges of operators. In particular, it can be used to affirm the conjecture of Halmos [8].

Theorem 3.5 (Chi and Li [4, Theorem 2.4]). *Let $A \in \mathcal{B}(\mathcal{H})$ be a contraction. Then*

$$\overline{W(A)} = \bigcap \{\overline{W(U)} : U \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}) \text{ is a unitary dilation of } A\}.$$

(*In the finite-dimensional case, the closure signs on the numerical ranges can be omitted.*)

We conclude this section with some related problems and show that Theorem 3.2 does not admit further generalizations.

As mentioned in the Introduction, each contraction $A \in \mathcal{B}(\mathcal{H})$ has a unitary power dilation U. This leads to important consequences such as the von Neumann inequality. However, the analogous statement for the constrained unitary power dilation is not valid as shown in the following simple example [4, Example 5.3].

Example 3.6. Let A be the zero matrix in M_n. Then obviously $A + A^* \le O$ and A is contractive. However, there is no unitary dilation U of A such that $U + U^* \le O$ and

$$U^k = \begin{pmatrix} A^k & * \\ * & * \end{pmatrix}, \qquad k = 1 \text{ and } 2.$$

Proof. Suppose $A = O$ has a unitary dilation $U = \begin{pmatrix} O & C \\ B & D \end{pmatrix}$ with

$$U + U^* = \begin{pmatrix} O & B^* + C \\ B + C^* & D + D^* \end{pmatrix} \le O.$$

It follows that $C = -B^* \ne O$ and thus $U^2 = \begin{pmatrix} -B^*B & * \\ * & * \end{pmatrix}$ cannot have the zero matrix at the upper left corner. □

Theorem 3.2 shows that if $A \in M_n$ is a contraction subject to a *single* affine constraint, then A has a unitary dilation subject to the same constraint. It is natural to ask whether more constraints can be added to the statement above. The following example [4, Example 5.4] shows that in general it is not possible to find a normal dilation subject to *two* constraints.

Example 3.7. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then A is a contraction satisfying $-I \le A + A^* \le I$. However, A has no normal dilation N satisfying $-I \le N + N^* \le I$.

Proof. Suppose $N = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$ is a normal dilation of A such that $-I \le N + N^* \le I$. Since the leading 2×2 submatrix of $N + N^*$ is unitary, we deduce that $N + N^* = (A + A^*) \oplus H$. Thus, $B = -C^* \ne O$. It follows that $NN^* \ne N^*N$ by comparing the leading 2×2 submatrices on both sides. □

4 Joint spectral circles and unitary similarity orbits

For $\mu \in \mathbf{C}$ and $r \geq 0$, we write $\Gamma(\mu; r) = \{z \in \mathbf{C} : |z - \mu| = r\}$ for the circle centered at μ with radius r. (When $r = 0$, the degenerated circle is the singleton $\{\mu\}$.) Notably, each single operator $A \in \mathcal{B}(\mathcal{H})$ is associated with a canonical circle $\Gamma(\mu_0; r)$ as follows:

Lemma 4.1 (Choi and Li [6]). *For each operator $A \in \mathcal{B}(\mathcal{H})$, there is a unique pair of $(\mu_0, r_0) \in \mathbf{C} \times [0, \infty)$ so that $r_0 = \|A - \mu_0 I\| \leq \|A - \mu I\| \quad$ for every $\mu \in \mathbf{C}$.*

Proof. Assume that the above inequality is true for $\mu_0 = \mu_1$ and μ_{-1}. Then for $\tilde{\mu} = (\mu_1 + \mu_{-1})/2$, we have

$$2\|A - \tilde{\mu} I\|^2 \geq \|A - \mu_1 I\|^2 + \|A - \mu_{-1} I\|^2 \geq \|(A - \mu_1 I)^*(A - \mu_1 I) + (A - \mu_{-1} I)^*(A - \mu_{-1} I)\|$$

$$= \|2(A - \tilde{\mu} I)^*(A - \tilde{\mu} I) + \frac{|\mu_1 - \mu_{-1}|^2}{2} I\| = 2\|A - \tilde{\mu} I\|^2 + |\mu_1 - \mu_{-1}|^2/2;$$

it follows that $\mu_1 = \mu_{-1}$ as desired. □

Remark 4.2. If A is a normal operator, the canonical optimal circle $\Gamma(\mu_0; r_0)$ as determined in Lemma 4.1 is the circle, with minimum radius, enclosing the spectrum of A; i.e.,

$$r_0 = \min_{\mu \in \mathbf{C}} \max\{|\alpha - \mu| : \alpha \in \sigma(A)\} = \max\{|\alpha - \mu_0| : \alpha \in \sigma(A)\}.$$

In other words, the optimal circle is the unique circle Γ to enclose the whole spectrum $\sigma(A)$ subject to the additional condition:
(**)*The center of Γ lies in the convex hull of $\Gamma \cap \sigma(A)$.*

Specifically, if A is a normal operator with a finite spectrum, then we will need only to consider finitely many circles Γ arising from any one of the following two types:
(1) Each pair of two points of $\sigma(A)$ determine the diameter of a circle Γ.
(2) Each acute-angle triangle with three vertices from $\sigma(A)$ determines a circle Γ passing through the vertices.
Among all circles of these two types, the optimal circle is the only circle to enclose the whole spectrum $\sigma(A)$.

To see the full significance of the spectral circles, we need the optimal normal dilations. Recall that every contraction in $B(H)$ has a unitary dilation. Applying affine transformations, we see that if $A \in B(H)$, $\mu \in \mathbf{C}$ and $r \geq 0$ satisfy $\|A - \mu I\| \leq r$, then A has a normal dilation $\tilde{A}$ such that $\sigma(\tilde{A}) \subseteq \Gamma(\mu; r)$.

Proposition 4.3 (Choi and Li [6, Section 3.2]). *Suppose $A \in B(H)$. Then*

$$\begin{aligned} & \sup\{\|A - U^*AU\| : U \text{ is unitary }\} \\ = \ & \min \sup\{\|\tilde{A} - \tilde{U}^*\tilde{A}\tilde{U}\| : \tilde{U} \text{ is unitary in } (\mathcal{H} \oplus \mathcal{H})\}, \end{aligned}$$

where min *is taken over all possible normal dilations $\tilde{A}$ of A acting on the larger Hilbert space $\mathcal{H} \oplus \mathcal{H}$. Moreover, let $\mu_0 \in \mathbf{C}$ be such that*

$$\|A - \mu_0 I\| \leq \|A - \mu I\| \quad \text{for every } \mu \in \mathbf{C},$$

and let $r_0 = \|A - \mu_0 I\|$. *Then each normal dilation* $\tilde{A}$ *of* A *so that* $\sigma(N) \subseteq \Gamma(\mu_0; r_0)$ *satisfies*

$$2r_0 = \sup\{\|A - U^*AU\| : U \text{ is unitary}\} = \sup\{\|\tilde{A} - \tilde{U}^*\tilde{A}\tilde{U}\| : \tilde{U} \text{ is unitary}\}.$$

Without referring to normal dilations, we can re-state Prop. 4.3 as follows:

Theorem 4.4 (Choi and Li [6, Section 3.2]). *Let* $A \in B(H)$, *and let* $\mu_0 \in \mathbf{C}$ *be such that*

$$\|A - \mu_0 I\| \le \|A - \mu I\| \quad \text{for all } \mu \in \mathbf{C}.$$

Set $r_0 = \|A - \mu_0 I\|$. *Then*

$$2r_0 = \sup\{\|A - U^*AU\| : U \text{ unitary}\}$$

and

$$\|f(A) + U^*g(A)U\| \le \max_{z\in\Gamma(\mu_0;r_0)} |f(z)| + \max_{z\in\Gamma(\mu_0;r_0)} |g(z)|$$

for each unitary U *and each pair of polynomials* $f(z)$ *and* $g(z)$.

Note that the equality $2r_0 = \sup\{\|A - U^*AU\| : U \text{ unitary}\}$ can be viewed as the optimal case of the last inequality with $f(z) = z - \mu_0$ and $g(z) = \mu_0 - z$.

We can further extend the above discussion to two operators $A, B \in B(H)$ and obtain the following theorem concerning their joint spectral circles in connection with the distance between their unitary similarity orbits.

Theorem 4.5 (Choi and Li [6, Theorem 3.3]). *Let* $A, B \in B(H)$, *and let* $\mu_0 \in \mathbf{C}$ *be such that*

$$\|A - \mu_0 I\| + \|B - \mu_0 I\| \le \|A - \mu I\| + \|B - \mu I\| \quad \text{for all } \mu \in \mathbf{C}.$$

Set $r_1 = \|A - \mu_0 I\|$ *and* $r_2 = \|B - \mu_0 I\|$. *Then*

$$\sup\{\|A - U^*BU\| : U \text{ unitary}\} = r_1 + r_2 \tag{4.1}$$

and

$$\|f(A) + U^*g(B)U\| \le \max_{z\in\Gamma(\mu_0;r_1)} |f(z)| + \max_{z\in\Gamma(\mu_0;r_2)} |g(z)| \tag{4.2}$$

for each unitary U *and each pair of polynomials* $f(z)$ *and* $g(z)$.

Note that (4.1) can be viewed as the equality case of (4.2) for $f(z) = z - \mu_0$ and $g(z) = \mu_0 - z$.

The next proposition gives a description for the set of complex numbers μ_0 in the statement of Theorem 4.5.

Proposition 4.6 (Choi and Li [6, Prop.3.4]). *Let* $A, B \in B(H)$, *and let* $S(A, B)$ *be the set of complex numbers* μ_0 *satisfying*

$$\|A - \mu_0 I\| + \|B - \mu_0 I\| \le \|A - \mu I\| + \|B - \mu I\| \quad \text{for all } \mu \in \mathbf{C}.$$

Then $S(A, B)$ *is a compact convex set which is either a singleton or a line segment.*

Remark 4.7. In case A and B are normal operators, the evaluation of $S(A,B)$ is much related to the geometrical positions of $\sigma(A)$ and $\sigma(B)$. In particular, writing $A = A_1 + iA_2$, we can estimate the norm of A by means of the joint optimal spectral circles for the pair (A_1, iA_2). The following may be the most important result in the structure theory of operator norm computation.

Theorem 4.8 (Choi and Li [5, Prop. 2.5]). *Suppose A and B are self-adjoint operators subject to $a_1 I \le A \le a_2 I$ and $b_1 I \le B \le b_2 I$. Assume further that $a_2 \ge |a_1|$ and $b_2 \ge |b_1|$.*

(i) *If $a_1 b_2 + a_2 b_1 \ge 0$, then*

$$\|A + iB\| \le |a_2 + ib_2| = \sqrt{a_2^2 + b_2^2}.$$

(ii) *If $a_1 b_2 + a_2 b_1 \le 0$, then*

$$\|A + iB\| \le \tau + \tau',$$

where

$$\tau = |a_1 - z_0| = |a_2 - z_0| = \frac{1}{2}\sqrt{(a_1 - a_2)^2 + (b_1 + b_2)^2}$$

and

$$\tau' = |ib_1 - z_0| = |ib_2 - z_0| = \frac{1}{2}\sqrt{(a_1 + a_2)^2 + (b_1 - b_2)^2}$$

with $z_0 = \{(a_1 + a_2) + i(b_1 + b_2)\}/2$.

(iii) *The bounds in* (i) *and* (ii) *are sharp in the following sense: If $\{a_1, a_2\} \subseteq \sigma(A)$ and $\{b_1, b_2\} \subseteq \sigma(B)$, then there exists a unitary U such that $\|A + iU^*BU\|$ attains the upper bound.*

Suppose $\tilde{A}$ and $\tilde{B}$ are normal dilations of A and B. We have

$$\sup\{\|U^*AU - V^*BV\| : U, V \text{ unitary}\} \le \sup\{\|\tilde{U}^*\tilde{A}\tilde{U} - \tilde{V}^*\tilde{B}\tilde{V}\| : \tilde{U}, \tilde{V} \text{ unitary}\};$$

i.e., the distance between the unitary orbits of A and B is not larger than that of their normal dilations. Nevertheless, the following theorem shows that there always exist appropriate normal dilations whose unitary orbits are not farther apart.

Proposition 4.9 (Choi and Li [6, Prop. 3.5]). *Suppose $A, B \in B(H)$. Then*

$$\begin{aligned} & \sup\{\|U^*AU - V^*BV\| : U \text{ and } V \text{ are unitaries}\} \\ = \; & \min \sup\{\|\tilde{U}^*\tilde{A}\tilde{U} - \tilde{V}^*\tilde{B}\tilde{V}\| : \tilde{U} \text{ and } \tilde{V} \text{ are unitaries}\}, \end{aligned}$$

where min *is taken over all possible normal dilations $\tilde{A}$ and $\tilde{B}$ of A and B on the common Hilbert space $\mathcal{H} \oplus \mathcal{H}$. Moreover, let $\mu_0 \in \mathbf{C}$ be such that*

$$\|A - \mu_0 I\| + \|B - \mu_0 I\| \le \|A - \mu I\| + \|B - \mu I\| \quad \text{for every } \mu \in \mathbf{C},$$

$r_1 = \|A - \mu_0 I\|$, and $r_2 = \|B - \mu_0 I\|$. Then the set

$$\begin{aligned} \mathcal{C} = \{(\tilde{A}, \tilde{B}) : \; & \tilde{A} \text{ and } \tilde{B} \text{ are normal dilations of } A \text{ and } B \text{ on a common} \\ & \text{Hilbert space } \mathcal{H} \oplus \mathcal{H} \text{ with } \sigma(\tilde{A}) \subseteq \Gamma(\mu_0; r_1) \text{ and } \sigma(\tilde{B}) \subseteq \Gamma(\mu_0; r_2)\} \end{aligned}$$

is non-empty, and every pair $(\tilde{A}, \tilde{B}) \in \mathcal{C}$ *satisfies*

$$r_1 + r_2 = \sup\{\|U^*AU - V^*BV\| : U \text{ and } V \text{ are unitaries}\}$$
$$= \sup\{\|\tilde{U}^*\tilde{A}\tilde{U} - \tilde{V}^*\tilde{B}\tilde{V}\| : \tilde{U} \text{ and } \tilde{V} \text{ are unitaries in } \mathcal{B}(\mathcal{H} \oplus \mathcal{H})\}.$$

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