REPORT ON THE 2007 GREAT PLAINS OPERATOR THEORY SYMPOSIUM (GPOTS 2007)

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1. Overview

GPOTS 2007, the 27th such conference, was held in Lincoln, Nebraska, from May 15–20, 2007. We summarize the participant demographics and program here. For the complete list of participants, abstracts, open questions submitted by participants, copies of some lectures, and timetable see `http://www.math.unl.edu/~jorr/gpots07`, or `http://www.gpots.org`. (The site `http://www.gpots.org` is intended to be the permanent archive, but is currently off-line. We will maintain `http://www.math.unl.edu/~jorr/gpots07` as a mirror site until `www.gpots.org` becomes permanently on-line.)

1.1. Participants.

There were one hundred participants, who fell into the following categories.

Category	*Number*
Graduate Students	27
Post Docs	10
Nontenured Faculty	15
Tenured Faculty	46
Unemployed or Retired	2
Total	100

Sixteen of the participants were women. Also, there were 12 foreign participants: 4 from Canada, 3 from India, 2 from Venezuela and one each from Mexico, Ireland, and England.

1.2. Program.

There were 14 plenary lectures, each of which was either 55 minutes or 30 minutes. The 9 longer lectures were given by:

Hari Bercovici, Indiana University;
David Blecher, University of Houston;
Kenneth Davidson, University of Waterloo;
Marius Junge, University of Illinois at Urbana–Champaign;
Elias Katsoulis, East Carolina University;
Scott McCullough, University of Florida;
Vladimir Peller, Michigan State University;
Christopher Phillips, University of Oregon;
Dimitri Shlyakhtenko, University of California, Los Angeles.

The 5 shorter lectures were given by:

Victor Kaftal, University of Cincinnati;
David Kerr, Texas A&M University;
Gelu Popescu, University of Texas-San Antonio;
David Sherman, University of California–Santa Barbara;
Nik Weaver, Washington University in St. Louis.

There were 49 contributed 20-minute lectures, given in two parallel tracks. Of these, 9 were given by graduate students, 9 by postdocs, and 6 by women.

2. Recent Results and Future Challenges

As part of this report, we were asked to comment on recent, significant results and future challenges in Operator Theory/Operator Algebras. Gathering views from a broad selection of conference participants seem essential to capturing the full range of activity. Thus, we held an hour-long "round table" discussion on major achievements and future challenges in the field. All participants were invited to attend the discussion. Prior to the discussion, participants were asked to consider providing their responses to the following three questions at the discussion:

(1) What are recent significant results in your area(s) of interest?
(2) What do you see as important future challenges for this area?
(3) What "broader impacts" can result from your area? (This could include applications to other areas of math, or the more traditional NSF "broader impacts.")

The discussion was lively and, in the course of it, several participants volunteered to provide short summaries of activities in specific areas. In addition, we had already asked the plenary speakers to consider providing similar summaries.

What follows are some of the participants' thoughts and the short summaries. We have not attempted to rank the various areas or the specific questions and results mentioned by participants, and the summaries are presented in alphabetical order by author. (Most of the summaries were written independently, and we have not tried to eliminate overlaps.)

Part of the success of GPOTS comes from taking a broad view of Operator Algebras and Operator Theory. We feel that a similarly broad view of the results and challenges in the area will best show the dynamic nature of the subject and its many connections with other parts of mathematics and other subjects.

Bercovici, Hari:

Classical intersection theory for subvarieties of Grassmann manifolds predicts the existence of subspaces which meet the subspaces in several flags in a specified way. I think it would be important to see whether such intersection theories exist in the Grassmannians associated with a factor of type II_1, and whether different factors lead to different results. These Grassmann manifolds are simply collections of orthogonal projections with a given trace. As a sample, consider projections $e \leq e', f \leq f', g \leq g'$ in such a factor such that $\tau(e) = \tau(f) = \tau(g) = 1/3$ and $\tau(e') = \tau(f') = \tau(g') = 2/3$; here τ is the normalized trace, and no other comparisons are required. One expects from classical intersection theory that there should exist a projection p satisfying $\tau(e \wedge p) \geq 1/6, \tau(f \wedge p) \geq 1/6, \tau(g \wedge p) \geq 1/6$ and $\tau(e' \wedge p) \geq 1/3, \tau(f' \wedge p) \geq 1/3, \tau(g' \wedge p) \geq 1/3$. Generically, the solution should not be unique (though it may be). This result is equivalent to the (relative) invariant subspace problem in a II_1 factor.

Blecher, David:

Here are a couple of questions in areas which I think are interesting.

Paul Muhly suggested in the panel discussion that 'noncommutative function theory' is an important topic, and I agree. Several questions concern the variant of noncommutative H^∞ introduced by Arveson under the name of 'subdiagonal algebras,' and recently intensively studied by several authors. The most important question concerns the possibility of weakening the definition of these algebras, to allow the inclusion of important examples from e.g. free probability theory, while at the same time still retaining some of the strong theoretical results.

On operator spaces: The theory of 'operator spaces', namely of subspaces of C^*-algebras with completely bounded maps as morphisms, is currently in a flourishing period, with many international conferences in the last couple of years devoted to this topic alone, or together with its interactions with harmonic analysis, free probability, quantum computing, noncommutative L^p-spaces, etc. We mention a random sample of fascinating directions/open problems from this theory. First, Arveson's recent solution of his famous problem dating to around 1970 of the existence of the noncommutative Choquet boundary, reopens the whole subject of 'noncommutative convexity', which had been somewhat blocked until now. In view of the importance of convexity, an intensive investigation of the noncommutative case seems of urgency. Second, Haagerup and Musat have recently solved an old conjecture of Effros and Ruan, concerning a Grothendieck inequality for C^*-algebras involving the 'operator space projective tensor product' and a symmetrized form of the Haagerup tensor product. This opens up the direction of applications and extensions of this new result. For example, it now becomes important to check the related question of the equivalence of these operator space tensor products at every matrix level. This may be restated as asking whether every 'jointly completely bounded' $B(H)$-valued bilinear map on a pair of C^*-algebras can be decomposed as a sum of a 'completely bounded' and a 'flip of a completely bounded' bilinear map. A third question we will mention is the following: are the one-sided M-ideals of Blecher, Effros, and Zarikian proximal? Even the case of these one-sided ideals in nonselfadjoint operator algebras of this question is open, and easily motivated by the importance of proximinality in the classical theory.

Fialkow, Lawrence:

One recent impact of operator theory concerns polynomial optimization. Since 2000, there has developed a subfield which studies the interconnections between real algebra (the structure of positive polynomials on semi-algebraic sets), optimization of polynomials over semi-algebraic sets, and the theory of moments. (This emerging area has been the subject of special workshops at Oberwolfach, Amsterdam, Minneapolis, CIRM (Marseille), etc.) The idea of solving the moment problem by representing positive polynomials as weighted sums of squares is very old. There is also a more recent interaction in the other direction. A celebrated result of K. Schmudgen (1991) solved the full moment problem on compact semi-algebraic sets and, surprisingly, showed how this leads to a representation for positive polynomials on these sets. My work with Raul Curto (based in operator theory) provides an analogous theory for the (more general) truncated moment problem. In 2000, Jean Lasserre used our main result on "flat extensions" of moment matrices as the stopping criterion in his new algorithm for polynomial optimization over semi-algebraic sets. Lasserre's work has spurred much of the recent interdisciplinary interactions cited above.

Helton, William:

An area of operator theory very much in full flourish is free analysis. There are several branches but the ones we focus on in this exposition have some contact with linear systems engineering. One of the main developments in this area starting the 1990's was that of Matrix Inequalities and Linear Matrix Inequalities LMIs. Deciding which problems convert to LMIs is an extremely challenging question which is essentially one of functional analysis requiring the development of a real algebraic geometry for free algebras. This is an open ended and new subject where there is lots of progress.

Another area stems from recently established connections between operator algebras and multi-dimensional linear systems. The classical version of one such result is that any Schur class function (a holomorphic function on the unit disk with values in the closed unit disk) can be realized as the transfer function of a conservative discrete-time input/state/output linear system. Recent multi-variable generalizations of this result include a realization theory for Schur-Agler class functions over various types of domains in multidimensional Euclidean space (e.g., the ball, the polydisk or a domain with matrix polynomial defining function) as the transfer function of an appropriate type of conservative multidimensional linear system. A noncommutative analogue is the realization theory for an element of the free semigroup algebra (or more generally, a function in noncommuting indeterminates which is contractive on an operator-tuple domain with noncommuting polynomial defining function) as the transfer function of a corresponding conservative input/state/output system with evolution along a tree. There also now is a more general class of H-infinity algebras arising from a Fock space construction involving a W*-correspondence and a *-representation of a W*-algebra which encodes time-varying linear systems as one particular type of example. One can expect that the algebras associated with higher rank graphs and product decompositions along more general semigroups will have connections with other kinds of input/state/output linear systems. The engineering implications for these more exotic types of linear systems remain to be worked out.

Also in the area of analytic function theory there is a developing theory for free algebras. For example, now one has noncommutative pluriharmonic functions and many of the basics such as maximum principles and Poison representations are now known to hold in a free algebra.

Junge, Marius:

The theory of operator algebras, which has its roots in Murray-von Neumann's work in the 1930's and 1940's, has seen several new trends arise. All of these trends lead to new connection to other fields in mathematics. These trends are also a sign of renewal of the theory which show that the theory of operator algebras is a dynamic field of research. Here are a few examples.

(1) Tomita-Takesaki theory led to Connes's classification of hyperfinite factors; this work brought new insights to the field, based on concepts in mathematical physics.
(2) The classification program of nuclear C^*-algebras is an ongoing line of research for the last thirty years. This theory has deep connections to the classifications of groups and dynamical systems.
(3) Jones' subfactor theory revealed deep connection to knot theory and has recent applications to quantum computing.
(4) Voicluescu's free probability provides a new perspective of operator algebras in probabilistic terms and has deep connections with the theory of large random matrices and combinatorics.
(5) The theory of non-selfadjoint algebras has profound connections to questions in single operator theory and dynamical systems.
(6) The theory of operator spaces intertwined metric aspects from Banach space theory with operator algebras and lead to Pisier's solution of Halmos' problem. Operator space theory has some applications to quantum probability and also has interactions with quantum information theory.
(7) Popa's new approach to the fundamental group of a von Neumann algebra reveals surprising new connections to rigidity theory known in group theory.

Katsoulis, Elias:

In 1989 Popescu introduced a multivariable analogue of the unilateral forward shift. Since then the study of various multivariable shifts and their associated non-selfadjoint operator algebras has been the source of exciting new developments in operator algebra theory. Davidson and Pitts and Arias and Popescu introduced the non-commutative Toeplitz algebras and made important connections with multivariable function theory. Davidson, Katsoulis and Pitts introduced the free semigroup algebras whose study led, among others, to non-commutative versions of Marshall's theorem. Katsoulis and Kribs and later Davidson and Katsoulis established nest representation theory as an indispensable tool in the study of operator algebras, with their study of subalgebras of Cuntz-Krieger algebras.

In this conference, this trend continued with a wealth of exciting developments. Davidson and Katsoulis used multivariable shifts to create new operator algebras that encode the dynamics of iterated function systems; their theory makes connections with the theory of analytic varieties. Duncan clarified the situation regarding the C*-envelope of some related algebras using free product constructions. Power, in joint work with Solel, presented a class of algebras which generalizes the semicrossed products of the irrational rotations and whose classification makes connections with algebraic geometry. Popescu introduced a non-commutative Berezin transform. Further, Davidson, Power, and Dilian Yang obtained, for rank 2 graphs on a single vertex, that certain representations have a unique dilation to $*$-representations, a classification of atomic $*$-representations, and and a decomposition of C*-algebra for those graphs which are aperiodic. There is definitely a lot of activity with important connections with other fields of mathematics.

Kerr, David:

Combinatorial independence in dynamics

Since its inception in the first half of the twentieth century, measurable dynamics has been closely linked with the analysis of group representations on Hilbert spaces, both from the internal structural viewpoint and in applications to operator algebras. The key notion underlying many basic dynamical concepts like ergodicity and weak mixing is that of probabilistic independence, which in this context can be expressed in purely operator-theoretic terms.

It has only be much more recently that topological dynamics has been studied in relation to analytic phenomena in an analogous manner. In this case the connection is to Banach space geometry and the types of combinatorial analysis that have a played an important role in its study. It turns out that mixing properties are reflected in the appearance of ℓ_1 in different ways along orbits of functions. The link between entropy and ℓ_1 was discovered by Eli Glasner and Benjamin Weiss in the early 1990s and pursued by Hanfeng Li and myself with the crucial aid of Voiculescu's notion of approximation entropy from operator algebras. We have developed more generally a systematic approach to various kinds of mixing behavior (entropy, sequence entropy, untameness) based on combinatorial independence and using combinatorial arguments, which have the major advantage of portability.

Very recently we have shown that the idea of combinatorial independence can also be applied to measurable dynamics. In this case we ask that independent behavior be observable whenever we hide from view a small portion of the space. By combining probabilistic and combinatorial arguments, we carry out a fine-scale local analysis of entropy production for measure-preserving actions of any discrete amenable group. In particular, we obtain local characterizations of the maximal zero entropy and zero sequence entropy factors in terms of ℓ_1 structure, combinatorial independence, and Voiculescu's von-Neumann-algebraic approximation entropy.

One goal for the future is to reframe the theory of combinatorial independence in measurable dynamics in terms of tensor products so that it may be extended to noncommutative operator algebras. We have already indicated how this can be done in the topological setting and have established some results in this direction.

Muhly, Paul:

Advances in Noncommutative Function Theory

At the conclusion of the introduction to *Functions of Several Noncommuting Variables* (Bull. Amer. Math. Soc. 79 (1973)), Joseph Taylor ended with the apology: "Although we feel that the ideas presented here are promising, it is too early to predict whether or not a significant theory will result from further development." The "ideas" to which he was referring concerned a function theory and a concomitant functional calculus for operators that are based on the free algebra on a finite number of generators. Only in recent years has it become clear how prescient his ideas were. Nowadays, there are several avenues of development which address issues he initiated. Some come from the free analysis that underlies free probability and noncommutative limit laws. Some come from multiparameter systems theory, as discussed by Helton above. Others are driven by dilation theory and efforts to understand the role that commutivity plays in that theory. Others, still, are inspired by the problem of interpretting the representation theory of rings and modules, i.e., finite dimensional algebra, in the context of infinite dimensional Hilbert spaces. And of course, multiparameter spectral theory, which was Taylor's primary inspiration, has grown considerably over the years.

One particular focus has involved a number of different, but closely related, analogues of the algebra of bounded analytic functions on the open unit disc, H^∞. These include the free semigroup algebras and noncommutative disc algebras that were featured in the pioneering work of Popescu and of Davidson and Pitts. They also include completions of complex path algebras of graphs, or quivers, that have received a lot of attention in the work of Power, Kribs, Katsoulis, and others. And a comprehensive perspective for all these algebras was developed by Muhly and Solel under the name of Hardy algebras of W^*-correspondences.

The theory of Hardy algebras has raised an enormous array of problems and avenues for research. An especially piquant problem is that of developing a theory of "noncommutative spaces". The family of completely contractive, ultraweakly continuous, Hilbert space representations of a Hardy algebra has an analytic structure as an infinite dimensional Cartan domain, and the Hardy algebra may be realized on this domain as a class of bounded analytic operator-valued functions. However, not every bounded analytic function on the domain comes from an element in the Hardy algebra. Those that do have been characterized in terms certain types of kernels they determine - analogues of the (operator-valued) Pick kernels that have attracted so much attention in recent years. In fact, the work that has been done suggests that there may well be a noncommutative function theory based entirely on these Pick-like kernels that would subsume what is known in the commutative theory (i.e., the theory of reproducing kernel Hilbert spaces of analytic functions in one or more complex variables) and would include the function theory of Hardy algebras as well. The domains coming from representations of a Hardy algebra ought to be the building blocks for more general "noncommutative analytic manifolds or varieties". Examples from the theory of product systems over $\mathbb{R}_+$ and over $\mathbb{N}^k$ suggest that such a program may be feasible. Also, the work of Taylor mentioned above and variations on it more recently developed by Voiculescu should provide important tools. Further, examples coming from matrix inequalities and noncommutative algebraic geometry are likely to provide a rich supply of additional possibilities, tools and applications.

Muhly, Paul and Williams, Dana:

Advances in the theory of groupoids

Groupoids entered analysis through the work of Mackey when he encountered group actions in representation theory that had singular orbit spaces. He called certain equivalence classes of groupoids, arising through ergodic group actions, *virtual groups*. The idea is that a virtual group is a replacement for the non-existent stabilizing subgroup for a non-transitive ergodic group action. Groupoids have played an important role in group representation theory ever since. Subsequently, Connes used groupoids to study singular spaces that arise in foliation theory and groupoids have since played a vital role in geometry. Indeed, it is fair to say that groupoids nowadays are ubiquitous throughout those areas of mathematics where one may encounter quotient spaces for which no Borel cross sections to the quotient maps exist. Further, and perhaps more important, even when groupoids are not explicitly mentioned or applied in a problem from geometry or analysis, many of the techniques and perspectives in these fields are inspired by methodology that arises in groupoid analysis. Indeed, for example, much of the work on dynamical systems of the kind discussed in Phillips's contribution below has been inspired by groupoid methodology.

Groupoids, themselves, have a rich representation theory and harmonic analysis. This has developed to a large extent by generalizing the theory of crossed products obtained from groups acting on C^*-algebras, i.e., from C^*-dynamical systems. However, technology transfer has gone both ways and it is now fair to say that a symbiotic interaction between the theory of C^*-crossed products and groupoid C^*-algebras has developed.

A topic still of active research and the source of many problems for the foreseeable future is the fine structure of the primitive ideal space of a groupoid C^*-algebra. There has been considerable effort devoted to extending the so-called Mackey Machine, as modified by Echterhoff, Green, Rieffel, Sauvageot, and others, to groupoid C^*-algebras, the purpose of which is to identify the primitive ideal space and its topology. These efforts, in turn, have inspired work devoted to finding functorial descriptions of the dual notions of induction and restriction of representations in analogy with that for group actions and co-actions that have appeared in the literature. The model here is the path-breaking Memoir, *A categorical approach to imprimitivity theorems for C^*-dynamical systems*, Mem. Amer. Math. Soc. **180** (2006), no. 850, viii+169 pp., by Echterhoff, Kaliszewski, Quigg and Raeburn.

Peller, Vladimir:

I would like to mention the following important directions in my area:

Toeplitz operators with matrix-valued symbols. Many things that are well-known in the case of Toeplitz operators with scalar-valued symbols are unknown in the case of Toeplitz operators with matrix-valued symbols.

Schur multipliers. There are still many intriguing problems related to Schur multipliers. In particular, it is still unknown whether for $1 < p < 2$ a Schur multiplier of the Schatten-von Neumann class S_p must be completely bounded.

Phillips, N. Christopher:

Here are some problems and remarks concerning several areas. This discussion does not give sources for the theorems stated. Some are mine, some are joint work of myself and other people, and some are results I had nothing to do with at all.

1) Automorphisms of the Calkin algebra

Theorem 1.1. The existence of outer automorphisms of the Calkin algebra Q is undecidable in ZFC (Zermelo-Frankel set theory plus the Axiom of Choice).

The Continuum Hypothesis implies that there are outer automorphisms, but other axioms consistent with ZFC (assuming ZFC is consistent) imply that all automorphisms are inner.

The outer automorphisms constructed using the Continuum Hypothesis are all approximately inner, in fact, inner on every separable subalgebra.

The following two related questions remain open. In fact, the first is the original problem.

Problem 1.2. Does ZFC imply the nonexistence of $\alpha \in \mathrm{Aut}(Q)$ such that α_* is multiplication by -1 on $K_1(Q)$?

Problem 1.3. Does ZFC imply the nonexistence of $\alpha \in \mathrm{Aut}(Q)$ such that α_* is the identity on $K_1(Q)$ but such that α is not approximately inner?

2) Orbit equivalence of minimal actions on the Cantor set

Theorem 2.1. Let X be the Cantor set, and let $\mathbb{Z}^2$ act freely and minimally on X. The system $(\mathbb{Z}^2, X)$ is orbit equivalent to an AF equivalence relation.

This has been proved by Giordano, Matui, Putnam, and Skau, and they think they are close to getting the case of actions of $\mathbb{Z}^d$.

Problem 2.2. Again let X be the Cantor set, and let Γ be a countable amenable group. Let Γ act on X via a minimal essentially free action. Does it follow that the system (Γ, X) is orbit equivalent to an AF equivalence relation?

(For a minimal action, essentially free means that the set of fixed points of the action of each $\gamma \in \Gamma \setminus \{1\}$ is nowhere dense. If Γ is abelian, then minimal and essentially free implies free.)

3) Transformation group C*-algebras of minimal homeomorphisms

Let X be a compact metric space, let $h\colon X \to X$ be a minimal homeomorphism, and let $A = C^*(\mathbb{Z}, X, h)$. Suppose that the standard map $\rho_A\colon K_0(A) \to \mathrm{Aff}(T(A))$ has dense range. If X is finite dimensional (has finite covering dimension), it is known that A has tracial rank zero, and this is probably still true if h has mean dimension zero in the sense of Lindenstrauss and Weiss. (This is a strictly weaker condition than finite dimensionality of X.)

Problem 3.1. Does there exist any minimal homeomorphism h of a compact metric space X (necessarily infinite dimensional) such that ρ_A as above has dense range but A does not have tracial rank zero?

It would be really nice if mean dimension zero, or some related condition, turned out to be equivalent to A having tracial rank zero. However, there is little evidence for something like this.

Problem 3.2. Prove a classification theorem for stably finite C*-algebras which is strong enough to cover all transformation group C*-algebras of minimal homeomorphisms of finite dimensional compact metric spaces.

It is known that, in the smooth case (X a C^∞ manifold and h a C^∞ diffeomorphism), the transformation group C*-algebra is a direct limit, with no dimension growth, of recursive subhomogeneous C*-algebras. However, there are a number of examples, even in the smooth case, in which it has no nontrivial projections.

4) Transformation group C*-algebras of minimal actions

The ultimate goal is to understand the transformation group C*-algebra of a minimal and essentially free action of a countable amenable group Γ on a compact metric space X. (Here, "essentially free" is as in Section 2. For now, though, we would be very pleased if we just understood the free case.) For $\Gamma = \mathbb{Z}$, a lot is known, but for more complicated groups very little is known; not much is known even for actions of $\mathbb{Z}^2$.

Some of the earlier problems are special cases of later ones, but are listed anyway in the belief that they might be more tractable.

Problem 4.1. Suppose X is the Cantor set. Does it follow that $C^*(\Gamma, X)$ has real rank zero, stable rank one, and order on projections determined by traces?

This is known for $\Gamma = \mathbb{Z}^d$.

Problem 4.2. Suppose X is the Cantor set. Does it follow that $C^*(\Gamma, X)$ has tracial rank zero?

Some partial results are known for $\Gamma = \mathbb{Z}^d$.

If for some Γ, one can find even one minimal and essentially free action of Γ on the Cantor set such that the transformation group C*-algebra has tracial rank zero, then, for this particular Γ, it follows that $C^*(\Gamma)$ is quasidiagonal. Quasidiagonality of the C*-algebras of countable amenable groups is a long standing open problem about which little is known.

Comparison of these problems with Theorem 2.1 suggests the following problems:

Problem 4.3. Let X be the Cantor set, and let $R \subset X \times X$ be an equivalence relation carrying a topology making it an etale groupoid on X. Suppose R is affable, that is, there is some other etale topology on R such that the groupoid C*-algebra is AF. Does it follow that the groupoid C*-algebra of R, with its original topology, has tracial rank zero? Does it at least follow that this groupoid C*-algebra has real rank zero, stable rank one, and order on projections determined by traces?

5) Miscellaneous

The following three results are quite significant, but are by now several years old.

Theorem 5.1. Every simple higher dimensional noncommutative torus is an AT algebra.

Theorem 5.2. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Let G be one of $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, and $\mathbb{Z}/6\mathbb{Z}$, acting on A_θ as a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with its usual action. Then $C^*(G, A_\theta)$ is AF.

The proof of this one relies on both the Elliott classification program and known cases of the Baum-Connes Conjecture.

Theorem 5.3. The crossed product by the flip action of $\mathbb{Z}/2\mathbb{Z}$ of every simple higher dimensional noncommutative torus is an AF algebra.

Popescu, Gelu:

Comments on "Free analytic function theory" (my part of the story)

In the last two decades, significant progress has been made in noncommutative multivariable operator theory regarding dilation theory, its applications to interpolation in several variables, and unitary invariants for n-tuples of operators. More recently, a theory of holomorphic functions in several noncommuting (free) variables was initiated to provide a framework for the study of arbitrary n-tuples of operators. This theory enhances Popescu's program to develop a free analogue of Sz.-Nagy–Foias theory, for row contractions. A free analytic functional calculus was introduced and studied in connection with Hausdorff derivations, noncommutative Cauchy and Poisson transforms, and von Neumann inequalities. Several classical results from complex analysis have free analogues in the noncommutative multivariable setting. We mention here Cauchy, Liouville, Schwartz, Weierstrass, and Montel type results for free analytic functions. This study was extended to free pluriharmonic functions on noncommutative balls. The main tools used in this study are certain noncommutative transforms, introduced by Popescu, which generalize the classical transforms of Berezin, Poisson, Fantappie, Herglotz, and Cayley. Along this line, we mention some significant results such as a characterization of bounded free pluriharmonic functions in terms of multi-Toeplitz operators on Fock spaces, a Dirichlet type extension problem, a Herglotz-Riesz representation type result, and a Caratheodory interpolation type result for free holomorphic functions with positive real parts. Several other classical results from complex analysis are expected to have free analogues in this noncommutative multivariable setting. As recent results by Popescu suggest, this area of research could extend from ball-like domains to more general noncommutative domains (and subvarieties) of n-tuples of operators.

This area of research has potential applications to function theory in several complex variables and noncommutative algebraic geometry. The interplay between the noncommutative harmonic analysis and operator theory is expected to have applications to systems theory (in electrical engineering), prediction, scattering theory, operator algebras, function theory, and interpolation in several variables. Moreover, the expected results will also have potential applications in fields such as geophysics, image processing, and control theory.

Sherman, David:

Here are a few paragraphs about trends and recent advances in "relative operator theory," which I define as the investigation of operator theoretic questions for elements in a von Neumann algebra $\mathcal{M}$. Although the treatment of ring elements as operators is a basic theme in the subject, there has not always been sustained interest in questions whose principal motivations come from operator theory itself. These generalizations may be done for their own sake (i.e. stronger theorems, new tools in von Neumann algebras), or they may meet a specific demand (e.g. the II_∞ versions of Fredholm theory and spectral flow required for noncommutative geometry). Often it is the *unsuccessful* transition which leads to the most interesting mathematics.

There has been recent progress in the relative version of the invariant subspace problem [Dykema, Haagerup, Schultz]. The basic question is whether, for any $x \in \mathcal{M}$, there must be a nontrivial projection $p \in \mathcal{M}$ such that $xp = pxp$. Perhaps the most interesting result is that in a II_1 factor, such a p must exist whenever the support of the Brown measure of x is more than a single point. Attempts to construct an operator with no such p have failed, but yielded better understanding of certain "random" operators arising in free probability.

The Schur-Horn theorem states that for a self-adjoint matrix, the spectrum majorizes the diagonal (Schur); also, when one n-tuple of real numbers majorizes another, there is an $n \times n$ matrix having one as spectrum and one as diagonal (Horn). It is desired to extend this result to other von Neumann algebras, where one replaces "restriction to a diagonal" with "conditional expectation onto a maximal abelian $*$-subalgebra." For $B(\ell^2)$ and an atomic MASA, the theorem does not generalize without some care, and it is still not fully understood there [Arveson, Kadison, Kaftal, Neumann, Weiss]. The other focus is the Horn theorem for II_1 factors. Argerami and Massey have proved some approximative results, but the main issue is being actively considered by several researchers. There are applications to frame theory [Antezana, Massey, Ruiz, Stojanoff].

Finally, there has been some renewed interest in approximate unitary equivalence. This is a ubiquitous tool in the the classification of C^*-algebras, but the requirement that the target algebra be a von Neumann algebra allows for a wider range of questions. Following partial results of Kamei from the 1980s (reproved by other authors), simple characterizations of approximate equivalence have been pushed quite far in the past few years by Ding-Hadwin and Sherman. It is known since a 1998 example of Hadwin that the the theory does not generalize fully – that is, the analogue of Voiculescu's theorem does not hold for $*$-homomorphisms of C^*-algebras into von Neumann algebras. Understanding this obstruction (and proving partial results) is a worthy goal. It was announced by Ng at GPOTS 2007 that, based on work with Giordano, a certain kind of Voiculescu theorem holds for a type III factor if and only if the factor is injective.

Shlyakhtenko, Dimitri:

Voiculescu's free probability theory has very strong connections with the theory of random multi-matrix models and (as was shown by Voiculescu and others) provides the appropriate language to describe their limiting behavior. In particular, the setting of operators on a Hilbert space gives the necessary analytical framework to be able to pose and discuss analysis questions. There is a number of exciting directions in this area, one being the connection between combinatorial problems of enumeration of planar maps and large-N expansions in random matrix theory (this subject is going back to 't Hooft, Zuber and others in the physics literature, and has been made rigorous by A. Guionnet and E. Maurel-Segala). I reported on a joint work with A. Guionnet that has made use of free stochastic analysis to use operator-algebra results (such as lack of projections in the reduced C^*-algebra of a free group) to prove random matrix results (connectedness of limiting spectra of certain types of random matrix models). It turns out that the kind of analysis that goes into the study of free SDEs is very related to L^2-cohomology, and can be used to complete the computation of free entropy dimension for certain classes of groups.

Smith, Roger:

A very old problem in von Neumann algebras concerns the fundamental group $\mathcal{F}(M)$ of a finite factor M. This is a subgroup of the multiplicative positive real numbers that reflects whether M is isomorphic to a corner pMp for some projection $p \in M$, or more generally, to a corner of $M \otimes \mathbb{M}_n$ for some matrix algebra $\mathbb{M}_n$. For a long time, it was only known that this group could be $(0, \infty)$ or countable in some cases (due to Connes's work with Property T groups). Remarkable recent work of Sorin Popa established that *all* subgroups of $(0, \infty)$ can occur as fundamental groups, establishing this concept as an important isomorphism invariant and classification tool.

Another important problem was to know whether the outer automorphism group $\mathrm{Out}(M)$ of a finite factor could be trivial, saying that the only automorphisms arise from unitaries in the algebra. Popa, in joint work with two of his students Ioana and Peterson, not only showed that this was possible, but gave examples where the fundamental group was trivial while $\mathrm{Out}(M)$ could be any separable compact abelian group.

These results do not occur in isolation but are part of a broader investigation of rigidity in finite factors, an important ingredient in the perturbation theory of von Neumann algebras.

Tomforde, Mark:

Higher rank graph C^*-algebras are C^*-algebras created from categorical/combinatorial objects known as k-graphs. Although a detailed theory of the structure of k-graph algebras is underway, one of the main obstacles of the theory is finding examples of higher rank graph C^*-algebras and identifying which C^*-algebras are associated with particular k-graphs. A recent accomplishment in this regard is the result of Pask, Raeburn, Rordam, and Sims which shows that A$\mathbb{T}$-algebras arise as C^*-algebras of certain 2-graphs, and these 2-graphs may be viewed as "rank-2 Bratteli diagrams". An important project for the future is to identify other examples of higher rank graph C^*-algebras to which the existing theory may be applied. In particular, it would be interesting to produce examples of C^*-algebras with exotic K-theories to test many current conjectures.

Recently ring theorists have developed a theory of Leavitt path algebras in which a K-algebra is created from a directed graph. These Leavitt path algebras are defined in a way similar to graph C^*-algebras, and while the results of both theories are similar, different techniques have been required to examine the structure in each case. More surprisingly, each theory has made nontrivial contributions to the other, and results for one class often have implications for the others. An interesting project for the future is to examine this interaction further, and investigate the relationship between graph C^*-algebras and Leavitt path algebras. In addition, it would be useful for C^*-algebra theory to identify which properties of graph C^*-algebras are C^*-algebraic in nature and which properties are algebraic in nature. This could have consequences for many particular examples of graph C^*-algebras, such as Cuntz algebras, AF-algebras, and Kirchberg algebras.

Weaver, Nik:

Interactions between C^*-algebra and set theory have led to the recent solution or partial solution of several old open problems in classical C^*-algebra. In 2004 Akemann and Weaver used a set-theoretic principle called "diamond" which goes beyond standard set theory to give a counterexample to an old problem of Naimark asking whether there are any nontrivial C^*-algebras with only one irreducible representation up to unitary equivalence (*Consistency of a counterexample to Naimark's problem,* Proc. Nat. Acad. Sci. USA 101 (2004), 7522-7525). Whether it is also consistent with standard set theory that there are no counterexamples remains open. More recently

Akemann and Weaver used the continuum hypothesis to construct nontrivial pure states on $\mathcal{B}(\mathcal{H})$, negatively answering old questions of Kadison/Singer and Anderson ($\mathcal{B}(\mathcal{H})$ *has a pure state that is not multiplicative on any MASA,* to appear in Proc. Nat. Acad. Sci. USA), and Phillips and Weaver used the continuum hypothesis to construct an outer automorphism of the Calkin algebra, whose existence had been a longstanding open problem which originally arose in C^*-algebraic K-theory (*The Calkin algebra has outer automorphisms,* to appear in Duke Mathematical Journal). Weaver's brilliant undergraduate student Eric Wofsey has also used set-theoretic forcing arguments to partially answer a question of Hadwin about order-isomorphism of maximal chains of projections in the Calkin algebra ($P(\omega)$/*fin and projections in the Calkin algebra,* to appear in Proc. Amer. Math. Soc.).

This line of research has attracted interest in the set theory community, and the set theorist Ilijas Farah has completed the Phillips/Weaver result by showing that it is also consistent with standard set theory that all automorphisms of the Calkin algebra are inner (*All automorphisms of the Calkin algebra are inner,* manuscript, posted on the web). Weaver wrote a survey (*Set theory and* C^**-algebras,* Bull. Symb. Logic 13 (2007), 1-20) which is likely to attract further interest from this direction.

At GPOTS 2007 Weaver spoke on recent joint work with Greg Kuperberg that makes deep connections between operator algebras and quantum information theory. This work is based on a notion of "quantum metric" which is inherent in the theory of quantum error correction and in particular is used there to establish bounds on the capacity of quantum information channels relative to the degree of corruption to which they are immune. Some metric concept is essential here in order to measure the distance between a corrupted message and the original, so that one can speak of degrees of corruption.

Kuperberg and Weaver have generalized this quantum metric notion to a broad von Neumann algebra context, where it synthesizes and extends a substantial body of previous work on quantum metrics by Connes, Rieffel, and Weaver. It also provides a setting where classical and quantum error correction are no longer merely analogous, but are realized as special cases of a single theory. This theory treats the general case of "hybrid" information which is partly classical and partly quantum, a development which is likely to be important in practical applications of quantum information theory.