THE KADISON-SINGER QUESTION FOR TYPE II_1 FACTORS AND RELATED QUESTIONS

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REQUEST TO THE AUDIENCE: I will raise a large number of questions during this lecture. I don't claim expertise on all the quesitons raised, and I suspect there will be people in the audience that know much more than I do about some of them. Please speak up if you have something to add or see something that looks wrong.

I will view the Kadison Singer question in the following form. Let M be the von Neumann algebra direct sum of complex matrix algebras of all sizes,

$$M = \sum_{n=1}^{\infty} \oplus M_n.$$

The elements of M are sequences of matrices, the nth element of the sequence is an n x n matrix, and the norms of the sequence is the supremum of the norms of the matrices (which must therefore be finite).

Let D represent all the elements of M that are diagonal matrices in each coordinate. Then M is a finite von Neumann algebra and D is a maximal abelian C*-subalgebra of M (i.e. a MASA).

The Kadison-Singer question asks, "Do the pure states of D have unique state extension to M?"

Of course the question of unique extension of pure states from a subalgebra to a larger algebra can be asked in many contexts. My purpose here is to suggest a very natural context and to explain the relationship between that new context and the original K-S context. I will view M as a finite von Neumann algebra of type I, and keep the finite part for the new context. I will consider type II instead of type I von Neumann algebras. The easiest case is surely when the larger algebra is a factor, so let's look at that case.

NOTATION: N is a type II_1 factor and A is a MASA of M.

TYPE II_1 K-S QUESTION: Do the pure states of A have unique extensions to pure states of N?

Before I tell you what I know about this question, let's back up a bit and follow the lead of the Kadison-Singer paper. One of the questions they resolve is:

THEOREM: There is a unique conditional expection (i.e. projection of norm 1) from M onto D. This

result is particularly easy since D is generated by its 1-dimensional projections.

In the type II_1 case, however, there are no 1-dimensional projections, so the best we can say at this point is:

FACT: There is a unique **normal** conditional expectation $E : N \to A$.

QUESTION: Can there be non-normal conditional expections (CEs) from N onto A? As Kadison and

Singer have pointed out, the existence of two or more distinct CEs will quickly imply that there are pure states for which state extension is not unique from A to N.

PARTIAL ANSWER: If A has a separable predual (i.e. if it can be represented on separable Hilbert space), then the answer is "yes". Several key ideas in the proof of this result come from the incomparable Sorin Popa of UCLA, the world's leading expert on type II_1 factors.

Before I get to the proof, let me explain why this question is relevant to the original Kadison-Singer question. The algebra M has many ideals. Let I be a maximal proper two sided ideal of M. Long ago it was shown that the quotient C*-algebra M/I is actually a type II_1 factor. Further, the image of D in the quotient space is still a MASA. If there were two different CEs from M/I onto D/I, there would be pure states of D/I that do not have unique extensions to M/I. It is then routine to show that there are pure states of D that do not have unique extension to M, and then the answer to the K-S problem is "NO".

I know that N/I does not have a separable predual, and I believe that D/I does not have a separable predual, although I can't find that result in the literature. If you prove otherwise, please remember me fondly when you write up the negative solution to the K-S Question.

Now let's see how to prove: THEOREM: If A has a separable predual, then there is a singular CE from

N onto A.

COMMENT:While the proof given relies on separability at a key point, it is unknown whether or not there are non-separable counterexamples. I remind you that if separability is an unnecessary hypothesis, the the answer to the K-S question is "no". PROOF OUTLINE: Begin with the trace τ on N. Create a

sequence f_n of normal states of N such that $f_n|_A = \tau|_A$, and such that the support projections of the f_n tend to 0 strongly in N. Let g be any limit point of the f_n for the weak* topology on the state space of N.

Now use the separability of the predual of M to find a sequence $\{p_n\}$ of projections in A that generate N as a von Neumann algebra. Then recursively define a sequence $\{g_n\}$ of states of N as follows.

$g_1 = p_1 g p_1 + (1 - p_1) g (1 - p_1)$

$g_2 = p_2 g_1 p_2 + (1 - p_2) g_1 (1 - p_2)$

...

$g_n = p_n g_{n-1} p_2 + (1 - p_n) g_{n-1} (1 - p_n)$

...

It is pretty clear that $\{p_j : j = 1, ..., n\}$ will all lie in the centralizer of g_n. That means that for any $b \in N$, $g_n(p_j b) = g_n(b p_j)$. Further, it is also easy to check that $g_n|_A = \tau|_A$. Let h be a limit point of the $\{g_n\}$ in the weak* topology. Then h will have all of the $\{p_n\}$ in its centralizer, and $h|_A = \tau|_A$.

Now use h to construct a CE from N onto A by the following trick. $\{a\tau : a \in A\}$ is norm dense in A_*, the predual of A. Define a map Q from $\{a\tau : a \in A\}$ to the dual N^* of N by $Q(a\tau) = ah$. Show that this is isometric (because $h|A = \tau|_A$). Thus Q extends to a linear isometry of A_* into N^*. It dual, Q^*, takes N^{**} to A. Considering N as canonically embedded into N^{**}, we restrict Q^{**} to the embedded copy of N, and thus get a norm 1 linear map G from N into A.

Next we show that G is a CE. This will follow from the fact that a weak* dense subset of A lies in the centralizer of h.

Finally, recall that there is a unique normal CE from N onto A. Thus, if G is not normal, then we have found two distinct CE's from N onto A, and the theorem is proved. We show that G is singular in the following sense. For any nonzero projection $p \in N$, there is a nonzero projection p_0 in N such that $p_0 \leq p$ and $G(p_0) = 0$. This is about as far from normal as you can get. This final fact follows from Takesaki's singularity criterion for states because our construction will ensure that the state h is singular. It is exactly here that the existence of a countable family $\{p_n\}$ of projections that generate A as a von Neumann algebra plays a key role.

COROLLARY: There is no countable set $\{q_n\}$ of projections in A that can be used to pave every element of M. I.e. there is no sequence of projections in D that will pave every element of M

EXERCISE: Find a different proof of this Corollary.

Recall from the proof outline that there were two processes that created sequences of states from which we selected an arbitrary weak* limit point. To explain them we need to use the unique normal conditional expectation

$E : N \to A$. The first starts with the trace and then defines $f_1 = 2r_1\tau$, where r_1 is any projection in N such that $E(r_1) = 2^{-1}1$, where 1 denotes the identity of A. Continuing in this way we take $f_n = 2^n r_n \tau$, where r_n is any projection in N such that $E(r_1) = 2^{-n}1$. There are many choices of the projections $\{r_n\}$, and consequently there are many singular states g on N such that $g|_A = \tau|_A$. (QUESTION: How many?). While countability appears in this process, it is not essential to the outcome of finding a singular state of N that restricts to the trace on A. (QUESTION: Is there a pure state of N that restricts to the trace on A? PARTIAL ANSWER: I think the answer is "yes" if $N = M/I, A = D/I$.)

The next process begins with one of the many singular states g that restricts to the trace on A. We then use the Kadison-Singer method (that they borrowed from von Neumann, "On rings of operators III", 1940) of building a "diagonal process", which, in this case, is just a CE from N onto A. For the record, von Neumann stated in his 1940 paper that diagonal processes "deserve to be studied for their own sake." Therefore I need not apologize for doing a bit of that here.

DEFINITION: Define a directed set W as follows. Elements of W will be finite subsets of projections in A with sum 1. Given two such subsets F, G, we say $F \geq G$ if F is a refinement of G (i.e. each element of F is dominated by an element of G). For $F \in W, g \in N^*, b \in N$ define $g_F \in N^*$ by $g_F = \sum_{p\in F}(pgp)$ and $b_F = \sum_{p\in F}(pbp)$. Note that this definition implies that $g(b_F) = g_F(b)$.

REMARK: For any $b \in N, F_1, F_2 \in W, F_1 \leq F_2, p \in F_1, g \in N^*$, we easily see that $b_{F_2}p = pb_{F_2}$ and p is in the centralizer of g_{F_2}.

THEOREM: $\forall b \in N, \lim_F b_F = E(b)$ (strong topology).

COROLLARY: $\forall g \in N_*, \lim_F g_F = E_*(g|_A)$ (norm topology).

THEOREM: The following statements are equivalent (but remember that they are all **false** when A has separable predual).a. $\forall g \in N^*, \lim_F g_F = E^*(g|_A)$ (weak* topology). b. Each pure state of A has unique state extension to N. c. For every $b \in N$, $\lim_F \|b_F - E(b)\| = 0$. d. If Z denotes the set of pure states of N that restrict to pure states on A, then $\forall g \in Z, \lim_F g_F = E^*(g|_A)$ (weak* topology). e. For every $b \in N$, $\lim_F b_F = E(b)$ for the $\sigma(N, N^*)$ topology.

Now let's see what we don't know. There is a lot of that, so let me raise some specific questions. Let X denote the set of all singular states of N that restrict to the trace on A. QUESTION: Is the existence of a non-normal CE from N onto A equivalent to the existence of a pure state of A with non-unique state extension to N?

Returning to the original K-S problem:

QUESTION: If J is any non-trivial ideal of M, is there a unique CE from M/J onto D/J? If not, then K-S = "no".

Back to N and A.

QUESTIONS: Is the (N, A) analog of Joel's theorem true? I.e. if f_0 is a pure state of A and if $f = f_0 \circ E$, is f a pure state of N? And by the way, how can we describe some pure states of N, and maybe even some that are pure when restricted to A?

It is tempting to believe that since A is a continuous von Neumann algebra (i.e. no minimal projections); thus Kadison and Singer showed that there were many CEs in that case. However, their argument (as well as subsequent proofs by Joel and perhaps others) only works when you start with $B(H)$, for H separable. I don't know of a proof for non-separable H. In any case, $B(H)$ is much bigger than N, so CEs that are distinct on $B(H)$ may not be distinct on N.

WE DO KNOW SOMETHING!!!!

We know that when A has separable predual, then all of the following statements are FALSE.

(a). $\forall g \in N^*, \lim_F g_F = E^*(g|_A)$ (weak* topology). (b). Each pure state of A has unique state extension to N. (c). For every $b \in N$, $\lim_F \|b_F - E(b)\| = 0$.

This means that there are examples for which each statement fails.

QUESTION: What are some of these examples?

We know that statement (a) holds when g is a normal functional. In fact, we know that convergence in norm holds. To find a singular g for which the statement fails, we will need the Axiom of Choice. The same hold for any example for which statement (b) fails. However, statement (c) is quite familiar to folks who have studied the original Kadison-Singer question. In that context, it is just paving, and in that context there are many results showing that certain elements of M can be paved. However, in the (N, A) context, there exist elements of N that **can't** be paved. If we can find one, then MAYBE that example will lead to an element of M that can't be paved. Or maybe it will lead to a proof that examples of that kind can't exist in the (M, D) world. Or we could be led to pavings of more operators in M that were previously known. Until we find one, we can't say.

Factors of type II_1 with separable predual have been very extensively studied. There are uncountably many examples. Further, unlike the (M, D) world, where all MASAs are unitarily equivalent, your typical type II_1 factor has MASAs that are not equivalent and that have different properties. Let's look at a specific situation.

Let F_2 denote a free group on the genetators u, v. Let N be the corresponding group von Neumann algebra, and let A be the subalgebra generated by the free subgroup F_u generated by u. We know that for some elements $b \in N$, $\lim_F \|b_F - E(b)\| \neq 0$. We know that N lives inside $l^2(F_2)$, so $b = \sum_{w \in F_2} \lambda_w w$, where $\sum_{w \in F_2} |\lambda_w|^2$ is finite. We know that $E(b) = \sum_{n=-\infty}^{n=\infty} \lambda_{u^n} u^n$. Let's get more specific.

Suppose that $b = \sum_{n \neq 0} \lambda_{v^n} v^n$. I.e. b lives in a MASA that is orthogonal (even free) relative to A.

QUESTION: Does $\lim_F \|b_F - E(b)\| = 0$? (My guess is "yes". I think we need to be more clever that this in order to get a negative example. However, if my guess is right, it suggests that elements of MASAs that are orthogoal to D can be paved.)

EXERCISE: Compose (and answer!) an analogous question for the hyperfinite II_1 factor.

If time permits, here are some details of the proof that when A has separable predual, then there is a singular CE from N onto A.

LEMMA: $\tau|_A$ extends to a singular state on N. (POPA)

PROOF: Let $\{p_n\}$ be projections in N with $\tau(p_n) = 2^{-n}$ and $E_A(p_n) = 2^{-n}1$. (see Popa's "Orthogonal MASAs in matrix algebras"). Define a sequence of states of N by $f_n = 2^n \tau(\bullet p_n)$. Note that if $a \in A$, then $f_n(a) = 2^n \tau(a p_n) = 2^n \tau(E(a p_n)) = 2^n \tau(a E(p_n)) = 2^n \tau(a(2^{-n}1)) = \tau(a)$. Take f to be any limit point of $\{f_n\}$ in N^*. Since $f(\vee_n^\infty p_k) = 1$ for all n and $\tau(\vee_n^\infty p_k) \leq 2^{-n+1}$, we see that f is singular and extends the trace on A.

DEFINITION: Let $X = \{f \in N_1^+ : f singular, f(1) = 1, f|_A = \tau|_A\}$. For each $f \in X$ let A_f denote the intersection of A and the centralizer of f.

COROLLARY: X is weak* countably compact.

PROOF: Weak* cluster points of countable sets of singular functionals are also singular. This result actually appeared in my 1966 dissertation so Popa refers to it (with a smile) as "Akemann's Singularity

Criterion". However, it really is a very easy corollary of Takesaki's criterion. Here are some more details for those not familiar with singularity.

The dual space of any von Neumann algebra is a direct sum of the predual and the space of singular functionals. Takesaki proved that a singular state f is characterized by the criterion that for any nonzero projection p, there is a nonzero projection q such that $f(q) = 0$.

Thus for any countable family f_n of singular states, we set $h = \sum 2^{-n} f_n$. If f is a limit point of $\{f_n\}$, then $h(q) = 0$ implies $f(q) = 0$ for any projection q. I.e. f is singular.

PROPOSITION: For any $f \in X$, A_f is a strongly closed subalgebra of A.

PROOF: Let $f \in X$. Let $\{a_\alpha\}$ be a net in A_f that converges strongly to $a \in A$. To show that $a \in A_f$, it suffices to show that $(af - fa)(x) = 0$ for all $x \in N$. This follows immediately from the following computation as supplied by Popa.

$$|f(x(a_\alpha - a))| \leq f(xx^*)^{1/2} f((a_\alpha - a)^*(a_\alpha - a))^{1/2} =$$

$$f(xx^*)^{1/2} \|a_\alpha - a\|_2 \to 0.$$

Similarly,

$$|f((a_\alpha - a)x)| \leq f((a_\alpha - a)(a_\alpha - a)^*)^{1/2} f(x^*x)^{1/2} =$$

$$\|a_\alpha - a\|_2 f(x^*x)^{1/2} \to 0.$$

COROLLARY: If A_* is separable, then there is $g \in X$ such that $A_g = A$. (POPA)

PROOF: Let $\{p_n\}$ be a countable set of projections in A that generates A as a von Neumann algebra. For any $g_0 \in X$, recursively define $g_n = p_n g_{n-1} p_n + (1 - p_n) g_{n-1} (1 - p_n)$, and let g be a limit point of $\{g_n\}$. Then $g \in X$ by the last Corollary and $A_g = A$ by the last Proposition.

COROLLARY: If A_* is separable, then there is a singular CE from N onto A, so pure state extensions are not unique.