Metric (matrix norm) characterizations of several important classes of operators and operator spaces

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Joint work with Matthew Neal

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Abstract

We give some new characterizations, of unitaries, isometries, unital operator spaces, unital function spaces, function systems, operator systems, dual operator systems, dual function systems, C^*-algebras, and related objects.

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These characterizations only employ the vector space and operator space structure.

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- Good characterizations are totally fundamental
- Some of the objects we discuss have other characterizations in the literature

- Another good characterization:

Ruan's theorem: An operator space (i.e. subspace of $B(H)$) 'is' a vector space X with a norm $\| \cdot \|_n$ on $M_n(X)$ for all $n \in \mathbb{N}$, such that

(R1) $\|axb\|_n \leq \|a\|\|x\|_n\|b\|$, for all $a, b \in M_n$,

(R2) $\left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\|_{m+n} = \max\{\|x\|_m, \|y\|_n\}$.

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- Uses only 'internal structure'

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• All the characterizations presented in this talk are metric-linear: only in terms of the underlying operator space structure

that is, only using the vector space structure and the bare matrix norms

Operator spaces were ‘born’ in Arveson’s 1969 Acta paper (complete contractions, complete isometries, ...)

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His spaces though were usually at least unital operator spaces:

Unital subspaces of C^*-algebras: $1_A \in X \subset A$

Such spaces have played a significant role in operator space theory in and since Arveson 1969 (one reason being that they include most known nonselfadjoint operator algebras, operator systems, etc)

Despite Ruan's 1988 characterization of operator spaces, over the years there has been no abstract characterization of unital operator spaces

Here is our answer to this question:

Notation: $$u_n = \begin{bmatrix} u & 0 & \cdots & 0 \\ 0 & u & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & u \end{bmatrix}$$

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Theorem If X is an operator space, $u \in X$, then (X, u) is a unital operator space if and only if

$$\|[u_n \ \ x]\| = \left\| \begin{bmatrix} u_n \\ x \end{bmatrix} \right\| = \sqrt{2},$$

for all $x \in M_n(X)$ of norm 1, and all $n \in \mathbb{N}$.

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for all $x \in M_n(X)$ of norm 1, and all $n \in \mathbb{N}$.

- This condition also characterizes 'unitaries in X'

We say an element v in an operator space X is an isometry in X if there exists a complete isometry T from X into $B(K, H)$, for Hilbert spaces H and K, with $T(v)$ an isometry.

Similarly v is a unitary in X if there exists a complete isometry T from X into a C^*-algebra with $T(v)$ a unitary.

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Theorem If X is an operator space, $u \in X$, then u is an isometry in X iff $\left\| \begin{bmatrix} u_n \\ x \end{bmatrix} \right\| = \sqrt{2}$ holds for all $x \in M_n(X)$ of norm 1, and all $n \in \mathbb{N}$

Similarly for coisometries, unitaries

Better result for C^*-algebras or TROs ($Z \subset B(K,H)$ with $ZZ^*Z \subset Z$):

Theorem In a C^*-algebra or TRO A, we have $u^*u = 1$ if and only if $\left\| \begin{bmatrix} u \\ x \end{bmatrix} \right\| = \sqrt{2}$ holds for all $x \in A$ of norm 1

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Q: $\exists$ other 'metric-linear' characterization of unitaries in C^*-algebras? (not using functionals?)

Other characterizations of isometries?

Proof:

We prove that in a C^*-algebra or TRO, if $\|[u \quad x]\|^2 = 2$ for all $x \in A$ of norm 1, then $uu^* = 1$.

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Let $a = [u \;\; x]$, then the C^*-identity gives $\|a\|^2 = \|a^*a\| = 2$.

By writing a^*a as a diagonal matrix plus another matrix, get

$$\|a^*a\| \leq \max\{\|u^*u\|, \|x^*x\|\} + \|u^*x\| = 1 + \|u^*x\| \leq 2$$

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Hence $\|u^*x\| = 1$. That is, the operator L_{u^*} of left multiplication by u^* is an isometric right module map from A onto the closed right ideal (submodule) u^*A of A^*A. However every isometric C^*-module map is 'unitary'. That is, in C^*-module language,

$$\langle uu^*x, x\rangle = \langle u^*x, u^*x\rangle = \langle x, x\rangle = x^*x, \qquad x \in A.$$

Since $uu^* \leq 1$, it follows easily that $\|(1 - uu^*)x\|^2 = 0$ as desired. $\square$

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This is because it is easy to 'metrically' describe selfadjoint elements and positive elements. For example, as the:

u-hermitian elements x: $\exists$ constant K s.t. $\|u + itx\|^2 \leq 1 + Kt^2$ for all $t \in \mathbb{R}$.

u-positive contractions: u-hermitian, and $\|u - x\| \leq 1$

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A deeper result:

Theorem A unital operator space (X, u) (characterized above) is an operator system iff

$$\inf\left\{\left\|\left[\begin{array}{cc} tu & x \\ y & tu \end{array}\right]\right\| \ : \ y \in \mathrm{Ball}(X)\right\} \leq \sqrt{t^2 + 1}$$

for all $t \in \mathbb{R}$ and $x \in \mathrm{Ball}(X)$.

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A deeper result:

Theorem A unital operator space (X, u) (characterized above) is an operator system iff for all $x \in \mathrm{Ball}(X)$ there exists an element $y \in \mathrm{Ball}(X)$ with $\left\|\begin{bmatrix} tu & x \\ y & tu \end{bmatrix}\right\| \leq \sqrt{t^2+1}$ for all $t \in \mathbb{R}$. It is also equivalent to:

$$\inf\left\{\left\|\begin{bmatrix} tu & x \\ y & tu \end{bmatrix}\right\| \;:\; y \in \mathrm{Ball}(X)\right\} \leq \sqrt{t^2+1}$$

for all $t \in \mathbb{R}$ and $x \in \mathrm{Ball}(X)$.

- The original characterization of operator systems, due to Choi and Effros, was not in terms of the matrix norms, but in terms of the positive cone in $M_n(X)$, $\forall n$

Characterizing C^*-algebras

• It seems interesting to ask if you can recognize simply from the norms and the vector space structure, when a unital operator space is a C^*-algebra

• Closely related to the question of recovering a forgotten product on a C^*-algebra

• Since unitaries have been characterized above, to characterize C^*-algebras it essentially suffices to characterize when the product of every two unitaries u and v in X is again in X

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Theorem A unital operator space $(X, 1)$ (characterized above) possesses a product with respect to which it is isomorphic to a C^*-algebra via a unital complete isometry, if and only if X is spanned by the unitaries in X (characterized above), and for every unitary v in X we have

$$\inf\left\{\left\|\begin{bmatrix} t1 & y \\ z & tv \end{bmatrix}\right\| \ : \ z \in \mathrm{Ball}(X)\right\} \leq \sqrt{t^2+1}$$

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Recovering a forgotten product:

Lemma Suppose that $I_H \in X \subset B(H)$, and that v is a coisometry (resp. isometry) on H which lies in X. If $y, z \in \mathrm{Ball}(X)$ then $z = -vy^*$ (resp. $y = -z^*v$) in $B(H)$ if and only if $\left\| \begin{bmatrix} t1 & y \\ z & tv \end{bmatrix} \right\| \leq \sqrt{1+t^2}$ for all $t \in \mathbb{R}$

Characterizing dual operator systems

In our paper we raised the old question of whether weak* closed unital selfadjoint subspaces of $B(H)$ are precisely the operator systems which are also a dual operator space

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Later I asked Bojan Magajna the question, and indeed:

Theorem Weak* closed unital selfadjoint subspaces of von Neumann algebras are exactly the operator systems which are also a dual operator space

Characterizing unital function spaces and systems

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These are the unital function spaces spanned by their g-hermitians. Or:

Theorem Function systems are the 'selfadjoint' function spaces possessing a selfadjoint unitary

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Idea of proof: What needs to be done is to show that the normal states are dense in the states. $\square$

- The latter is a sufficient condition for any weak* closed unital selfadjoint subspace of a von Neumann algebra, and is also true for $B(H)$-valued completely positive maps

Some other results:

In our paper, we also discuss changing the unit in an operator system

Example: If u is any unitary in a C^*-algebra or TRO A, then (A, u) is an operator system.

Moreover, any two unitaries $u, v \in A$ induce in some sense the same operator system structure, since the map $T(x) = vu^*x$ on A is a surjective complete isometry taking u to v; and hence T is a complete order isomorphism too

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This fails badly for more general operator spaces. It is easy to find operator spaces X with unitaries u, v for which (X, u) is not an operator system but (X, v) is; or for which they are both operator systems but there exists no surjective complete isometry taking u to v. Moreover, the latter can be done with u and v inducing the same involution on X

We examine this phenomena and try to analyze what is going on. E.g.:

Proposition Let v be a unitary in an operator subsystem $X \subset B(H)$. Then (X, v) is an operator system and the involution associated with v equals the original involution, iff $v \in X_{\rm sa}$ and v is in the center of $C^*_{\rm e}(X)$