TOPOLOGICAL STABLE RANK OF BANACH ALGEBRAS

Kenneth R. Davidson

University of Waterloo

GPOTS, June, 2008

COAUTHORS

This is joint work with

- You Qing Ji (Jilin)
- Rupert Levene (Belfast)
- Laurent Marcoux (Waterloo)
- Heydar Radjavi (Waterloo)

K. DAVIDSON, R. LEVENE, L. MARCOUX AND H. RADJAVI
On the topological stable rank of non-selfadjoint operator algebras
Math. Annalen **341** (2008), 239–253.

K.R. DAVIDSON AND Y.Q. JI
Topological stable rank of nest algebras
preprint, 2008 on arXiv and www.math.uwaterloo.ca/∼krdavids/.

BACKGROUND

DEFINITION (COVERING DIMENSION)

X compact Hausdorff space.
$\dim X = n$: least integer $n \geq 0$ so that every finite open cover has refinement so no point is covered more than $n+1$ times.

EXAMPLE

$$\dim[0,1]^n = n.$$

THEOREM: TFAE

- $\dim X \leq n$.
- $\mathrm{C}(X, \mathbb{R}^{n+1} \setminus \{0\})$ is dense in $\mathrm{C}(X, \mathbb{R}^{n+1})$.
- $\forall \varepsilon > 0 \ \forall f_1, \ldots, f_{n+1} \in \mathrm{C}(X, \mathbb{R})$, there are $g_1, \ldots, g_{n+1} \in \mathrm{C}(X, \mathbb{R})$ with $\|f_i - g_i\| < \varepsilon$ so that the ideal $\langle g_1, \ldots, g_{n+1} \rangle = \mathrm{C}(X, \mathbb{R})$.

TOPOLOGICAL STABLE RANK

DEFINITION (RIEFFEL)

unital Banach algebra $\mathfrak{A}$
$Rg_n(\mathfrak{A}) = \{(a_1, \dots, a_n) : \sum_{i=1}^n a_i\mathfrak{A} = \mathfrak{A}\}$.
The right topological stable rank of $\mathfrak{A}$ is least integer $n = \operatorname{rtsr}(\mathfrak{A})$ such that $Rg_n(\mathfrak{A})$ is dense in $\mathfrak{A}^n$.
(i.e. right invertible $1 \times n$ matrices dense in $\mathcal{R}_n(\mathfrak{A})$.)

So $\operatorname{rtsr}(\mathrm{C}_{\mathbb{R}}(X)) = \dim X + 1$ over the reals. But working over $\mathbb{C}$,

PROPOSITION

$$\operatorname{rtsr}(\mathrm{C}(X)) = \left\lfloor \frac{\dim X}{2} \right\rfloor + 1.$$

PROPOSITION

If $\mathfrak{A}$ is a B-algebra, then*

$$\operatorname{rtsr}(\mathfrak{A}) = \operatorname{ltsr}(\mathfrak{A}) =: \operatorname{tsr}(\mathfrak{A}).$$

BASS STABLE RANK

DEFINITION (BASS)

unital ring A

$Lg_n(A) = \{(a_1, \dots, a_n) : \sum_{i=1}^n Aa_i = A\}$.

The left Bass stable rank of A is least integer $n = \mathrm{lBsr}(A)$ such that if $(a_i)_{i=1}^{n+1} \in Lg_{n+1}(A)$, then $\exists b_i \in A$ so that $(a_1 + b_1 a_{n+1}, \dots, a_n + b_n a_{n+1}) \in Lg_n(A)$.

THEOREM (VASERSTEIN)

For any unital ring A,

$$\mathrm{lBsr}(A) = \mathrm{rBsr}(A) =: \mathrm{Bsr}(A).$$

QUESTION (RIEFFEL)

Is $\mathrm{rtsr}(\mathfrak{A}) = \mathrm{ltsr}(\mathfrak{A})$ for all Banach algebras?

EVIDENCE

PROPOSITION (RIEFFEL)

$$\mathrm{lBsr}(\mathfrak{A}) \leq \mathrm{rtsr}(\mathfrak{A}).$$

So

COROLLARY

$$\mathrm{Bsr}(\mathfrak{A}) \leq \min\{\mathrm{rtsr}(\mathfrak{A}), \mathrm{ltsr}(\mathfrak{A})\}.$$

THEOREM (HERMAN–VASERSTEIN)

If $\mathfrak{A}$ is a C*-algebra, then $\mathrm{tsr}(\mathfrak{A}) = \mathrm{Bsr}(\mathfrak{A})$.

However

THEOREM (JONES–MARSHALL–WOLFF)

$$\mathrm{Bsr}(\mathcal{A}(\mathbb{D})) = 1 < 2 = \mathrm{tsr}(\mathcal{A}(\mathbb{D})).$$

EXAMPLES

THEOREM (RIEFFEL): TFAE

1. $\mathrm{rtsr}(\mathfrak{A}) = 1$
2. $\mathrm{ltsr}(\mathfrak{A}) = 1$
3. $\mathfrak{A}^{-1}$ is dense in $\mathfrak{A}$.

PROPOSITION

If $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ contains *isometries* u, v with $\mathrm{Ran}(u) \perp \mathrm{Ran}(v)$, then $\mathrm{rtsr}(\mathfrak{A}) = \infty$.

COROLLARY

$$\mathrm{tsr}(\mathcal{B}(\mathcal{H})) = \infty.$$

IDEALS AND QUOTIENTS

PROPOSITION (RIEFFEL)

If $\mathfrak{J} \lhd \mathfrak{A}$, *then* $\operatorname{rtsr}(\mathfrak{A}/\mathfrak{J}) \leq \operatorname{rtsr}(\mathfrak{A})$.

THEOREM (RIEFFEL)

If $\mathfrak{J} \lhd \mathfrak{A}$ and $\mathfrak{J}$ has a b.a.p., then $\operatorname{rtsr}(\mathfrak{J}) \leq \operatorname{rtsr}(\mathfrak{A})$.

THEOREM (RIEFFEL)

If $\mathfrak{J} \lhd \mathfrak{A}$, then $\operatorname{rtsr}(\mathfrak{A}) \leq \max\{\operatorname{rtsr}(\mathfrak{J}), \operatorname{rtsr}(\mathfrak{A}/\mathfrak{J}) + 1\}$.

Unfortunately, in n.s.a. algebras, most ideals do not have b.a.p.
We will compensate by assuming that the sequence splits.

COMPLETELY FINITE ALGEBRAS

DEFINITION

$\mathfrak{A}$ is completely finite if $\forall\ R = \begin{bmatrix} a_1 & \dots a_n \end{bmatrix}$ right invertible, $\exists$ invertible $A \in \mathfrak{M}_n(\mathfrak{A})$ with first row R.

PROPOSITION

Completely finite algebras are stably finite.

EXAMPLE

$\mathrm{C}(S^5)$ is stably finite but not completely finite.

THEOREM

If $0 \to \mathfrak{J} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{J} \to 0$ splits and $\mathfrak{A}/\mathfrak{J}$ is completely finite, then

$$\mathrm{rtsr}(\mathfrak{A}) = \max\{\mathrm{rtsr}(\mathfrak{J}), \mathrm{rtsr}(\mathfrak{A}/\mathfrak{J})\}.$$

NEST ALGEBRAS

DEFINITION

A nest $\mathcal{N}$ is a complete chain of subspaces containing $\{0\}, \mathcal{H}$.
A nest algebra $\mathcal{T}(\mathcal{N}) = \{T : TN \subset N, \ \forall N \in \mathcal{N}\}$.

EXAMPLES

1. $\mathcal{H} = \sum^{\oplus} \mathcal{H}_k$, $\dim \mathcal{H}_k = n_k$, $N_k = \sum_{i=1}^{k} \mathcal{H}_i$.
   This is an atomic nest of order type ω.
   $\mathcal{T}(\mathcal{N})$ are block upper triangular.
2. $\mathcal{H} = L^2(0,1)$, $N_t = L^2(0,t)$ for $0 \le t \le 1$.
   This is a continuous nest.
3. Take $\mathcal{N}$ any nest. $\mathcal{N}^\perp = \{N^\perp : N \in \mathcal{N}\}$.
   $\mathcal{T}(\mathcal{N}^\perp) = \mathcal{T}(\mathcal{N})^*$ are lower triangular w.r.t. $\mathcal{N}$.
   $\operatorname{ltsr}(\mathcal{T}(\mathcal{N}^\perp)) = \operatorname{rtsr}(\mathcal{T}(\mathcal{N}))$ and $\operatorname{rtsr}(\mathcal{T}(\mathcal{N}^\perp)) = \operatorname{ltsr}(\mathcal{T}(\mathcal{N}))$.

NEST ALGEBRAS

PROPOSITION

If $\mathcal{N}$ *contains* $N_1 \gneqq N_2 \gneqq N_3 \gneqq \dots$, *then* $\operatorname{rtsr}(\mathcal{T}(\mathcal{N})) = \infty$.

Proof.

Take quotient onto $\mathcal{T}(N_1 \ominus \bigcap N_k)$.

Build two isometries with orthogonal ranges.

COROLLARY

In the following cases (with separable $\mathcal{H}$), $\operatorname{rtsr}(\mathcal{T}(\mathcal{N})) = \infty$.

1. $\mathcal{N}$ is uncountable.
2. $\mathcal{N}$ is countable, but not of ordinal type.
3. $\mathcal{N}$ has an infinite dimensional atom.

NEST ALGEBRAS

Certain nest algebras provide counterexample to Rieffel's question:

THEOREM (DLMR)

If $\mathcal{N}$ has order type ω and $n_k \geq 4 \sum_{i<n} n_i$, then

$$\operatorname{rtsr}(\mathcal{T}(\mathcal{N})) = 2 \quad \text{and} \quad \operatorname{ltsr}(\mathcal{T}(\mathcal{N})) = \infty.$$

Sketch. Let $E_k = P_{N_k} - P_{N_{k-1}}$ be the atoms. rank $E_k = n_k$.
$P_{N_{k-1}} A E_k$ and $P_{N_{k-1}} B E_k$ are supported on "half" of E_k.
This provides room for the construction.

NEST ALGEBRAS

PROPOSITION

If $\mathcal{N}$ is a nest of order type ω with finite rank atoms, then $\mathcal{T}(\mathcal{N})$ is completely finite.

If $\mathcal{M} \subset \mathcal{N}$, then $\mathcal{T}_0(\mathcal{M}) \triangleleft \mathcal{T}_0(\mathcal{N}) \triangleleft \mathcal{T}(\mathcal{N}) \subset \mathcal{T}(\mathcal{M})$.

Moreover $0 \to \mathcal{T}_0(\mathcal{M}) \to \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N})/\mathcal{T}_0(\mathcal{M}) \to 0$ splits.

DEFINITION

Say $\mathcal{M} = \{N_{k_i} : i \geq 1\}$ is finite index in $\mathcal{N} = \{N_k : k \geq 1\}$ if $\sup k_{i+1} - k_i < \infty$.

COROLLARY

If $\mathcal{M}$ is finite index in $\mathcal{N}$, then $\operatorname{rtsr}(\mathcal{T}(\mathcal{M})) = \operatorname{rtsr}(\mathcal{T}(\mathcal{N}))$.

NEST ALGEBRAS

THEOREM

If for all $q \geq 1$, $\exists N_0 < N_1 < \cdots < N_q$ *in* $\mathcal{N}$ *with*

$$\dim(N_j \ominus N_{j-1}) \geq \dim(N_{j+1} \ominus N_j) \quad \textit{for} \quad 1 \leq j < q,$$

then $\operatorname{rtsr}(\mathcal{T}(\mathcal{N})) = \infty$.

DEFINITION

Let $\beta(\mathcal{N})$ be the sup of all q for which the hypotheses above hold.

THEOREM (DJ): TFAE

1. $\operatorname{rtsr}(\mathcal{T}(\mathcal{N})) = 2$.
2. $\operatorname{rtsr}(\mathcal{T}(\mathcal{N})) < \infty$.
3. $\beta(\mathcal{N}) < \infty$ and $\mathcal{N}$ has no infinite rank atoms.

SPECIAL CASES

COROLLARY (1)

If $\mathcal{N}$ *has ordinal type of* ω^2 *or larger, then* $\operatorname{rtsr}(\mathcal{T}(\mathcal{N})) = \infty$*.*

COROLLARY (2)

Suppose $\mathcal{N} = \{N_k : k \geq 1\}$ and n_k is monotone increasing. Let $d_j = |\{k : 2^{j-1} < n_k \leq 2^j\}|$. Then

1. If $\sup d_j = \infty$, then $\operatorname{rtsr}(\mathcal{T}(\mathcal{N})) = \infty$.
2. If $\sup d_j < \infty$, then $\operatorname{rtsr}(\mathcal{T}(\mathcal{N})) = 2$.

COROLLARY (3)

If n_k is monotone and $\liminf_{k\to\infty} n_k^{1/k} = 1$, then $\operatorname{rtsr}(\mathcal{T}(\mathcal{N})) = \infty$.

AN ALGORITHM

DEFINITION

$\mathcal{N} = \{N_k, \mathcal{H} : k \geq 0\}$, $\operatorname{rank}(N_k \ominus N_{k-1}) = n_k < \infty$.
$\mathcal{M} = \{N_{k_i}, \mathcal{H} : i \geq 0\} \subset \mathcal{N}$. $\mathcal{N}$ is *monotone relative to* $\mathcal{M}$ if

$$n_{k_{i-1}+1} \leq n_{k_{i-1}+2} \leq \cdots \leq n_{k_i} \text{ for } i \geq 1.$$

$\mathcal{M}$ is the *minimal relatively monotone subnest* of $\mathcal{N}$
if one also has $n_{k_i} > n_{k_i+1}$ for $i \geq 1$.

THEOREM

$\mathcal{N} = \{N_k, \mathcal{H} : k \geq 0\}$, $\operatorname{rank}(N_k \ominus N_{k-1}) = n_k < \infty$.
$\mathcal{N} \supset \mathcal{N}_1 \supset \mathcal{N}_2 \supset \ldots$ where $\mathcal{N}_{i+1}$ is the minimal relatively monotone subnest of $\mathcal{N}_i$.
If $\operatorname{rtsr}(\mathcal{T}(\mathcal{N})) < \infty$, then this sequence is finite.
If the sequence is finite, then $\operatorname{rtsr}(\mathcal{T}(\mathcal{N}))$ can be computed by using the monotone case and the split exact sequence theorem.