On sums of Hermitian operators in finite von Neumann algebras

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Eigenvalues of sums of Hermitian matrices

Suppose $A, B \in M_n(\mathbb{C})_{s.a.}$ have eigenvalues

$$\lambda_A(1) \geq \lambda_A(2) \geq \cdots \geq \lambda_A(n) \text{ of } A$$
$$\lambda_B(1) \geq \lambda_B(2) \geq \cdots \geq \lambda_B(n) \text{ of } B.$$

What can the eigenvalues of $A + B$ be?

Equivalently (in a symmetric reformulation), which triples

$$(\lambda_A(i))_{i=1}^n, \qquad (\lambda_B(j))_{j=1}^n, \qquad (\lambda_C(k))_{k=1}^n,$$

of sequences in $\mathbb{R}^n_{\geq}$ arise as eigenvalues of $A, B, C \in M_n(\mathbb{C})_{s.a.}$ with $A + B + C = 0$?

Some necessary conditions:

Since $A + B + C = 0$, taking the trace, we need

$$\sum_{i=1}^{n} \lambda_A(i) + \sum_{j=1}^{n} \lambda_B(j) + \sum_{k=1}^{n} \lambda_C(k) = 0. \qquad \text{(Tr)}$$

Using $C = (-A) + (-B)$, we have

$$\lambda_C(1) \leq \lambda_{-A}(1) + \lambda_{-B}(1) = -\lambda_A(n) - \lambda_B(n).$$

Thus, we also need

$$\lambda_A(n) + \lambda_B(n) + \lambda_C(1) \leq 0.$$

Other classic inequalities:

[H. Weyl, 1912], [K. Fan, '49], [B.V. Lidskii '50].

Horn's inequalities:

[A. Horn '62] considered inequalities of the form

$$\sum_{i\in I}\lambda_A(i)+\sum_{j\in J}\lambda_B(j)+\sum_{k\in K}\lambda_C(k)\leq 0, \qquad \text{(*IJK)}$$

where I, J, and K are subsets of $\{1,\ldots,n\}$ of the same cardinality.

He recursively defined sets T_r^n of such triples (I,J,K) with $|I|=|J|=|K|=r$.

Horn's conjecture: λ_A, λ_B, λ_C arise as eigenvalue sequences of A, B and C with $A+B+C=0$ if and only if

- the trace equality (Tr) holds
- the inequality (*IJK) holds for all $(I,J,K)\in\bigcup_{r=1}^{n-1}T_r^n$.

The sets T_r^n (Horn's definition)

Consider triples (I, J, K) of subsets of $\{1, \ldots, n\}$ with $|I| = |J| = |K| = r$

$$U_r^n = \left\{ (I, J, K) \middle| \sum_{i \in I} i + \sum_{j \in J} j + \sum_{k \in K} k = \frac{r(4n - r + 3)}{2} \right\}.$$

When $r = 1$, set $T_1^n = U_1^n$.

Otherwise, writing $I = \{i_1 < i_2 < \cdots < i_r\}$, etc., let

$$T_r^n = \\ = \left\{ (I, J, K) \in U_r^n \middle| \sum_{f \in F} i_f + \sum_{g \in G} j_g + \sum_{h \in H} k_h \geq \frac{p(4n - p + 3)}{2}, \right. \\ \left. \text{for all } p < r \text{ and } (F, G, H) \in T_p^r \right\}.$$

Using Schubert Calculus to prove Horn inequalities

Given a triple (I, J, K) how can we show that the Horn inequality

$$\sum_{i \in I} \lambda_A(i) + \sum_{j \in J} \lambda_B(j) + \sum_{k \in K} \lambda_C(k) \leq 0, \qquad (\text{*IJK})$$

holds whenever $A + B + C = 0$?

Find a projection P such that

$$\sum_{i \in I} \lambda_A(i) \leq \mathrm{Tr}(PAP), \qquad \sum_{j \in J} \lambda_B(j) \leq \mathrm{Tr}(PBP),$$

$$\sum_{k \in K} \lambda_C(k) \leq \mathrm{Tr}(PCP)$$

and use that $\mathrm{Tr}(PAP + PBP + PCP) = 0$.

To find a projection $P \in M_n(\mathbb{C})$ so that

$$\sum_{i \in I} \lambda_A(i) \leq \mathrm{Tr}(PAP), \qquad (*\mathrm{PAP})$$

write $I = \{i_1 < i_2 < \cdots < i_r\}$ and let $v_1, \ldots, v_n$ be orthonormal eigenvectors for A with $Av_i = \lambda_A(i)v_i$.

Consider the flag $\mathcal{E} = (E_m)_{m=0}^n$ where E_m is the rank m projection onto $\mathrm{span}\,\{v_1, \ldots, v_m\}$.

If

1. $\mathrm{rank}\,(P) = r$ and if
2. $\mathrm{rank}\,(P \wedge E_{i_\ell}) \geq \ell$ for all $1 \leq \ell \leq r$,

then (*PAP) holds.

The set of projections P satisfying 1. and 2. above is the *Schubert variety* $S(\mathcal{E}, I)$.

To prove that (*IJK) holds for a particular (I, J, K), it suffices to show that for all flags $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$, we have

$$S(\mathcal{E}, I) \cap S(\mathcal{F}, J) \cap S(\mathcal{G}, K) \neq \emptyset.$$

Proof of Horn's conjecture:

[Klyachko, Knutson, Tao, ...]:

To each $(I, J, K) \in U_r^n$, i.e. subsets of $\{1, \ldots, n\}$ so that $|I| = |J| = |K| = r$ and

$$\sum_{i \in I} i + \sum_{j \in J} j + \sum_{k \in K} k = \frac{r(4n - r + 3)}{2},$$

one associates a nonnegative integer $c_{IJK}^{(n)}$, which is the *Littlewood–Richardson coefficient* $c_{\alpha,\beta}^{\gamma}$ where α, β and γ are certain partitions of integers obtained from I, J and K, respectively.

The Littlewood–Richardson coefficients

- have a combinatorial description in terms of integer fillings of Young diagrams,
- appear in the representation theory of permutation groups and of general linear groups,
- arise in the cohomology ring of the Grassmanian $G(r, \mathbb{C}^n)$ (when multiplying Schubert cycles).

Moreover, from Horn's recursive definition one can show

$$T_r^n = \{(I, J, K) \in U_r^n \mid c_{IJK}^{(n)} > 0\},$$

- Many authors [1] showed that for every $(I, J, K) \in T_r^n$, the Horn inequality (*IJK) holds (whenever $A + B + C = 0$). This proves half of Horn's conjecture.
- [Klyachko '01] showed that the reverse direction would follow, i.e. the Horn inequalities together with the trace equality would determine the set of possible eigenvalues of A, B, C such that $A + B + C = 0$, if the *saturation conjecture* were known to hold, i.e.,

$$K \in \mathbb{N} \text{ and } c_{K\alpha,K\beta}^{K\gamma} > 0 \quad \Longrightarrow \quad c_{\alpha,\beta}^{\gamma} > 0.$$

- [Knutson, Tao '99] proved the saturation conjecture.
- [Belkale '01] showed that the (I, J, K) such that $c_{IJK}^{(n)} > 1$ are redundant.
- [Knutson, Tao, Woodward '01] give a direct proof of Horn's conjecture, and show that the set of Horn inequalties (*IJK) over all (I, J, K) with $c_{IJK}^{(n)} = 1$ is minimal.

[1] [Johnson '79], [Tataro '94], [Helmke and Rosenthal, '95], [Klyachko '01]

Analogue in a II_1–factor $\mathcal{M}$ with trace τ:

If "spectral data" of $a, b \in \mathcal{M}_{s.a.}$ are specified, what can be the "spectral data" of $a + b$?

Or, in symmetric form: what are the possible spectral data of a, b and c when $a + b + c = 0$?

"Spectral data":

$\mathrm{distr}(a) = \mu_a$, a Borel probability measure on $\mathbb{R}$, such that

$$\frac{\tau(a^k)}{\tau(1)} = \int_{\sigma(a)} t^k \, d\mu_a(t), \qquad (k \in \mathbb{N}).$$

We have the *eigenvalue function* $\lambda_a : [0, \tau(1)) \to \mathbb{R}$, given by

$$\lambda_a(t) = \sup\left\{ s \in \mathbb{R} \mid \mu_a((s, \infty)) > \frac{t}{\tau(1)} \right\}.$$

It is bounded, right–continuous, nonincreasing.

The eigenvalue function $\lambda_a : [0, \tau(1)) \to \mathbb{R}$ has properties expected of eigenvalues.

For example,

$$\tau(a^k) = \int_0^{\tau(1)} \lambda_a(t)^k \, dt.$$

Moreover, if $a \sim \mathrm{diag}(\alpha)$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_{\geq}$ and $\tau(1) = n$, then

$$\lambda_a(t) = \alpha_j, \qquad j - 1 \leq t < j, \quad j \in \{1, \ldots, n\}.$$

Analogues of Horn inequalities in a II_1–factor $\mathcal{M}$

If $(I, J, K) \in T^n_r$, we say that *the corresponding Horn inequality holds in* $\mathcal{M}$ if, taking the trace normalization $\tau(1) = n$, we have

$$\sum_{i \in I} \int_{i-1}^{n} \lambda_a + \sum_{j \in J} \int_{j-1}^{n} \lambda_b + \sum_{k \in K} \int_{k-1}^{k} \lambda_c \leq 0$$

whenever $a, b, c \in \mathcal{M}_{s.a.}$ and $a + b + c = 0$.

Connes's embedding problem

Does every II_1–factor $\mathcal{M}$ with separable predual embed in R^ω?

($R^\omega =$ the ultraproduct of the hyperfinite II_1-factor.)

Equivalent formulation: Does every element x of every II_1-factor $\mathcal{M}$ have matricial microstates?

Namely, given $m \in \mathbb{N}$ and $\gamma > 0$, is there k and $Y \in M_k(\mathbb{C})$ whose $*$–moments up to order m agree within distance γ to those of x:

$$\forall 1 \leq p \leq m \quad \forall \epsilon_1, \ldots, \epsilon_p \in \{1, *\},$$

$$|\mathrm{tr}(Y^{\epsilon_1} \cdots Y^{\epsilon_p}) - \tau(x^{\epsilon_1} \cdots x^{\epsilon_p})| < \gamma \ ?$$

Theorem. [Bercovici, Li, '06]: Let f, g and h be nonincreasing, right continuous, bounded functions from $[0,1)$ into $\mathbb{R}$. Then there are $a, b, c \in R^\omega$ (trace τ normalized so $\tau(1) = 1$) with $a + b + c = 0$ and

$$\lambda_a = f, \qquad \lambda_b = g, \qquad \lambda_c = h,$$

if and only if $\int_0^1 (f + g + h) = 0$ and

$$\sum_{i\in I} \int_{\frac{i-1}{n}}^{\frac{i}{n}} f + \sum_{j\in J} \int_{\frac{j-1}{n}}^{\frac{j}{n}} g + \sum_{k\in K} \int_{\frac{k-1}{n}}^{\frac{k}{n}} h \leq 0$$

for every $(I, J, K) \in \bigcup_{\substack{n\geq 2,\\ 1\leq r<n}} T_r^n$.

Corollary. If a II_1–factor $\mathcal{M}$ is embeddable in R^ω, then all Horn inequalities hold in $\mathcal{M}$.

Some Horn inequalities were known to hold in all II_1–factors:

[Bercovici, Li '01]: the analogues of the Freede–Thompson inequalities.

[Collins, D.]: the Horn inequalities for all $(I, J, K) \in T_3^n$ with $c_{IJK}^{(n)} = 1$.

Theorem. [Bercovici, Collins, D., Li, Timotin]: All Horn inequalities hold in all II_1–factors.

Proof: Practical Schubert Calculus

We need only consider those $(I, J, K) \in T_r^n$ with $c_{IJK}^{(n)} = 1$.

Let $\mathcal{M}$ be a II_1–factor with trace τ, normalized so that $\tau(1) = n$.

We consider flags $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ in $\mathcal{M}$, of the form

$$\mathcal{E} : \quad 0 = E_0 \leq E_1 \leq E_2 \leq \cdots \leq E_{n-1} \leq E_n = 1$$

with $\tau(E_j) = j$.

Goal: Solve the intersection problem by showing

$$S(\mathcal{E}, I) \cap S(\mathcal{F}, J) \cap S(\mathcal{G}, K) \neq \emptyset.$$

We must prove there is a projection $P \in \mathcal{M}$ such that $\tau(P) = r$ and

$$\tau(P \wedge E_{i_\ell}) \geq \ell, \qquad \tau(P \wedge F_{j_\ell}) \geq \ell, \qquad \tau(P \wedge G_{k_\ell}) \geq \ell$$

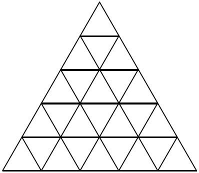
for all $1 \leq \ell \leq r$.

Our proof is constructive.

We describe an algorithm for constructing P from the projections in $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$, using lattice operations $\wedge$ and $\vee$. (But only in the case $c^{(n)}_{IJK} = 1$.)

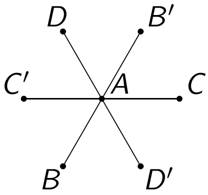
This is new, also in $M_n(\mathbb{C})$.

Outline: following [Knutson, Tao '99], triples $(I, J, K) \in T^n_r$ with $c^{(n)}_{IJK} = 1$ correspond to certain positive measures m supported on the edges of a triangular grid Δ_r



such that the small edges have integer masses, (shown here in the case $r = 5$).

The measures m must satisfy a balance condition: around every vertex



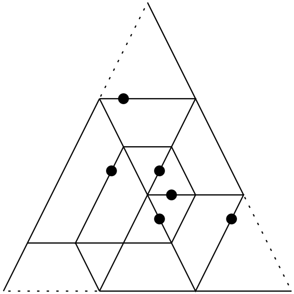
we must have

$$m(AB) - m(AB') = m(AC) - m(AC') = m(AD) - m(AD').$$

Then I, J, and K are read off from values of m on edges at the boundary of the triangle Δ_r.

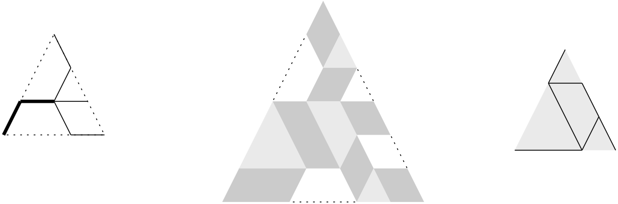
1. Decompose: Write the measure m as a sum of extremal ones, which are chacterized as being supported on *skeletons*.

For example (with $r = 6$):



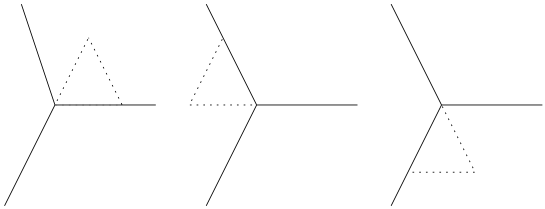
2. Reduce: It is possble to select one of these extremal measures μ_1 and to show that the intersection problem for *m* reduces to solving the intersection problems for μ_1 and for $m - \mu_1$ (which has fewer skeletons).

3. Dualize: The intersection problem for a measure μ is equivalent to the intersection problem for a dual measure μ^*. This duality (following [Knutson, Tao, Woodward '04]) is realized by *inflation* and *deflation*. For example:



If μ is an extremal measure supported on a skeleton, then μ^* is a sum of measures supported on skeletons having lower complexity.

4. The most reduced case:



corresponds to an intersection problem that has a trivial solution.

QED

We have shown that all Horn inequalities hold in all II_1–factors.

Can we conclude that all II_1–factors embed in R^ω?

For embeddability, given a pair (a_1, a_2) of self–adjoints in $\mathcal{M}$, we *need to find* microstates (A_1, A_2) in $M_n(\mathbb{C})$:

$$\mathrm{Tr}(A_{i_1} A_{i_2} \cdots A_{i_p}) \approx \tau(a_{i_1} a_{i_2} \cdots a_{i_p})$$

Using that the Horn inequalities hold in $\mathcal{M}$, we *can find* Hermitian matrices A_1 and A_2 such that

$$\lambda_{a_i} \approx \lambda_{A_i}\,, \qquad \lambda_{a_1+a_2} \approx \lambda_{A_1+A_2}\,.$$

A linearization of Connes's embedding problem.

Theorem. [Collins., D.] Suppose that whenever $\mathcal{M}$ is a II_1–factor, $a_1, a_2 \in \mathcal{M}_{s.a.}$, $\ell \in \mathbb{N}$ and $c_1, c_2 \in M_\ell(\mathbb{C})_{s.a.}$ there are $n \in \mathbb{N}$ and $A_1, A_2 \in M_n(\mathbb{C})_{s.a.}$ such that

$$\lambda_{a_i} \approx \lambda_{A_i}\,, \qquad \lambda_{c_1\otimes a_1 + c_2\otimes a_2} \approx \lambda_{c_1\otimes A_1 + c_2\otimes A_2}\,.$$

Then all II_1–factors embed in R^ω.

To attack this problem, we first need to understand the finite–dimensional situation.

Questions about the possible values of $\lambda_{c_1 \otimes A_1 + c_2 \otimes A_2}$:

- If we fix n, λ_{A_1}, λ_{A_2}, c_1 and c_2, is the set of all possible $\lambda_{c_1 \otimes A_1 + c_2 \otimes A_2}$ convex? This is the set

$$\{\lambda_{c_1 \otimes \mathrm{diag}(\lambda_{A_1}) + c_2 \otimes U \mathrm{diag}(\lambda_{A_2}) U^*} \mid U \in \mathcal{U}_n\}.$$

  No.

- If we fix n, λ_{A_1}, λ_{A_2}, c_1, c_2, and choose $d \in \mathbb{N}$ large enough, is the set

$$\{\lambda_{c_1 \otimes \mathrm{diag}(\lambda_{A_1}) \otimes I_d + c_2 \otimes U(\mathrm{diag}(\lambda_{A_2}) \otimes I_d) U^*} \mid U \in \mathcal{U}_{nd}\}$$

  convex?

- If we fix n, λ_{A_1}, λ_{A_2}, c_1 and c_2, is the set

$$\bigcup_{d=1}^{\infty} \{\lambda_{c_1 \otimes \mathrm{diag}(\lambda_{A_1}) \otimes I_d + c_2 \otimes U(\mathrm{diag}(\lambda_{A_2}) \otimes I_d) U^*} \mid U \in \mathcal{U}_{nd}\}$$

  convex?