The Spectrum of Groupoid C^*-algebras

Geoff Goehle

Dartmouth College

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Outline

Groupoids

Let G be a (topological) groupoid. We will assume that:

- G is locally compact Hausdorff.
- G is second countable.
- G has a Haar system.

Transformation Group Groupoids

Suppose H is a second countable LCH group acting on a second countable LCH space X.

Let $G = H \times X$.
Define groupoid actions on G via

$$(s,x)(t,s^{-1}\cdot x) = (st,x) \qquad (s,x)^{-1} = (s^{-1}, s^{-1}\cdot x)$$

Transformation group groupoids are *relatively* well behaved.

Groupoid C^*-algebras

Given a groupoid G we can build an associated (separable) C^*-algebra $C^*(G)$ using the standard "universal norm" construction.

- If G is a group then $C^*(G)$ is the usual group C^*-algebra.
- If $G = H \times X$ is a transformation groupoid then $C^*(G)$ is the usual transformation group C^*-algebra, $C^*(H, X)$.
- There are also lots of interesting examples which aren't based on transformation groups.

The Isotropy Subgroupoid

Given a groupoid G define the isotropy subgroupoid to be

$$S = \{\gamma \in G : r(\gamma) = s(\gamma)\}.$$

- We use $p = s|_S = r|_S$ to view S as a bundle over $G^{(0)}$.
- The fibres of S, denoted S_x, are the isotropy sub*groups* of G.
- S is a groupoid group bundle over $G^{(0)}$ with possibly varying fibres.

Note: S may not have a Haar system so we must assume that it does.

The Dual Isotropy Bundle

Suppose the isotropy subgroups of G are abelian so that S is an groupoid group bundle.

This implies that $C^*(S)$ is an abelian C^*-algebra and that $\widehat{S} := C^*(S)^\wedge$ is a second countable LCH space.

It turns out that $\widehat{S}$ is a groupoid group bundle and the fibre $\widehat{S}_x$ is the Pontryagin dual of S_x. [Muhly, Renault, Williams, 96]

Furthermore, G acts on $\widehat{S}$ via conjugation as follows. Given $\gamma \in G$ and $\omega \in \widehat{S}_{s(\gamma)}$

$$\gamma \cdot \omega(\eta) := \omega(\gamma^{-1}\eta\gamma)$$

for all $\eta \in S_{r(\gamma)}$.

The Induction Process

There is a method for inducing a representation of $C^*(S)$ to a representation of $C^*(G)$.

- Start with a representation ω of $C^*(S)$.
- Complete $C_c(S)$ into a Hilbert $C^*(S)$-module X.
- Find an action of $C^*(G)$ on X by adjointable operators.
- Use the Rieffel correspondence on ω and restrict to $C^*(G)$.
- The resulting representation is denoted $\operatorname{Ind}_S^G \omega$.

Inducing Irreducible Representations

Note: In general induction does not preserve irreducibility.

However, in this case one can show that $\operatorname{Ind}_S^G \omega$ is irreducible if ω is [RMW, 96; IW, 07] and therefore we can define a map

$$\phi : \widehat{S} \to C^*(G)^\wedge$$

where $\phi(\omega) := \operatorname{Ind}_S^G \omega$.

Induction is well behaved with respect to the hull-kernel topology and it's easy to see that ϕ is continuous.

Injectivity and Openness

The map ϕ is not injective, but with a little work one can show that given $\omega, \rho \in \widehat{S}$ we have:

$$\phi(\omega) = \phi(\rho) \quad \text{if and only if} \quad \exists\, \gamma \in G \text{ s.t. } \gamma \cdot \omega = \rho.$$

where G acts on $\widehat{S}$ via conjugation.

With significantly more work one can also show that ϕ is open.

Thus, ϕ factors to an continuous, open, injection

$$\tilde{\phi} : \widehat{S}/G \to C^*(G)^\wedge.$$

Surjectivity

Recall: G acts on $G^{(0)}$ as follows, for all $\gamma \in G$

$$\gamma \cdot s(\gamma) := r(\gamma).$$

It takes effort and ordinals (as well as the Mackey-Glimm dichotomy [Ramsay, 89]) but one can show that ϕ, and hence $\tilde{\phi}$, is surjective if $G^{(0)}/G$ is T_0.

Note: Assuming $G^{(0)}/G$ is T_0 is enough to make $C^*(G)$ GCR [Orloff, 06].

Where We Are

Theorem [G,07]

Suppose G is a second countable locally compact Hausdorff groupoid with Haar system. Furthermore, suppose that the isotropy subgroupoid S has abelian fibres and its own Haar system. If $G^{(0)}/G$ is T_0 then $\tilde{\phi} : \widehat{S}/G \to C^*(G)^\wedge$ is a homeomorphism.

Where We've Been

Suppose $G = H \times X$ is a transformation groupoid. Existing theorems [Williams, 81] allow one to compute the spectrum of $C^*(H, X)$ if H is abelian

Applied to transformation groups, my result only requires that the isotropy groups are abelian. *However*, we assumed that S had a Haar system and it's an unfortunate fact that this occurs if and only if the isotropy groups vary continuously (with respect to the Fell topology) [Renault, 91].

The action of $SO(3)$ on $\mathbb{R}^3 \setminus \{0\}$ is an example of a situation where the new result is useful.

Where We're Going

- Haar Systems
  - Requiring S to have a Haar system is a strong condition.
  - Can we generalize to "discontinuous" stabilizers?
- Beyond T_0
  - Everything except surjectivity can be extended to primitive ideals when $G^{(0)}/G$ is not T_0.
  - Can we extend surjectivity to the non-regular case?

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