Proper Actions of Groupoids on C^*-Algebras

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Proper Actions on Spaces

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If X is a free and proper G space then

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- It is very compelling to try to generalize the notion of proper action to dynamical systems (A, G, α) for which A in not necessarily commutative.

Proper Actions of Groups on C^*-algebras

Definition (Rieffel 1990)

Let α be an action of a LCH group G on a C^*-algebra A. We say the action is proper if $\exists$ a dense α-invariant $*$-subalgebra A_0 of A such that

1. *for any $a, b \in A_0$ the functions ${}_E\langle a, b\rangle(t) = \Delta(x)^{-1/2}a\alpha_t(b^*)$ and $t \mapsto a\alpha_t(b^*)$ are integrable.*
2. *$\forall\ a, b \in A_0\ \exists!$ element $\langle a, b\rangle_D \in M(A_0)^\alpha = \{d \in M(A) \mid dA_0 \subset A_0 \text{ and } \alpha_t(d) = d\}$ such that $\forall\ c \in A_0$ we have*

$$\int_G c\alpha_t(a^*b)dt = c\langle a, b\rangle_D$$

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Theorem (Rieffel 1990)

Let G act properly on a C^*-algebra A (via α) with respect to the dense subalgebra A_0. Then the generalized fixed point algebra for this action A^α is Morita equivalent to an ideal of the reduced crossed product.

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- Every Groupoid G has a unit space $G^{(0)}$ whose elements act trivially
- Every Groupoid G has maps $r, s : G \to G^{(0)}$ such that $r(\gamma)\gamma = \gamma$ and $\gamma s(\gamma) = \gamma$.

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- multiplication only makes sense if the end of the first path coincides with the beginning of the second.
- $G^{(0)}$ is the set of constant paths. Thus we can identify $G^{(0)}$ with X
- $r(\gamma) = \gamma(1)$ is the end of the path γ
- $s(\gamma) = \gamma(0)$ is the beginning of the path γ

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- This implies that $r, s : G \longrightarrow G^{(0)}$ are continuous.
- All Groupoids in this talk are assumed to be Locally Compact Hausdorff
- I will assume throughout this talk that each groupoid is endowed with a Haar system $\{\lambda^u\}_{u \in G^{(0)}}$

Upper Semi-Continuous C^*-Bundles

Definition (Upper Semi-Continuous C^*-Bundle)

An upper semi-continuous (usc) C^*-bundle $\mathscr{A}$ over X is a continuous map $p : \mathscr{A} \to X$ such that $A(x) := p^{-1}(x)$ is a C^*-algeba for every $x \in X$ and such that the norms of $A(x)$ vary upper semi-continously.

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Theorem (Hofmann, Dupré & Gillette)

There is a one to one correspondence between USC C^*-Bundles over X and $C_0(X)$-algebras, given by $A = \Gamma_0(X, \mathscr{A})$, the set of continuous sections of $\mathscr{A}$ vanishing at infinity.

Groupoid Dynamical Systems

Definition (Groupoid Dynamical System)

Suppose G is a groupoid and $\mathscr{A}$ is an Upper Semi-Continuous C^*-bundle over $G^{(0)}$ and $A = \Gamma_0(G^{(0)}, \mathscr{A})$, then an action of G on A is a family of $*$-isomorphisms $\{\alpha_\gamma\}_{\gamma \in G}$ such that

- $\alpha_\gamma : A(s(\gamma)) \to A(r(\gamma))$
- $\alpha_\gamma \alpha_\eta = \alpha_{\gamma\eta}$ *where this makes sense*
- *the map* $(\gamma, a) \mapsto \alpha_\gamma(a)$ *is continuous on* $G * \mathscr{A}$

Definition of Proper Actions

Definition (B 2008)

Let α be an action of a LCH groupoid G on a USC C^*-bundle $\mathscr{A}$ over $G^{(0)}$. Let $A = \Gamma_0(G^{(0)}, \mathscr{A})$. We say the action is **proper** *if $\exists$ a dense $*$-subalgebra A_0 of A such that*

1. *for any $a, b \in A_0$ the function* $_E\langle a, b\rangle(\gamma) = a(r(\gamma))\alpha_\gamma(b(s(\gamma))^*)$ *is integrable.*
2. $\forall\ a, b \in A_0\ \exists!$ *element* $\langle a, b\rangle_D \in M(A_0)^\alpha$ *where*

   $$M(A_0)^\alpha = \{d \in M(A) \mid dA_0 \subset A_0 \text{ and } \alpha_\gamma(d(s(\gamma))) = d(r(\gamma))\}$$

   such that $\forall\ c \in A_0$ *we have*

   $$\int_G c(r(\gamma))\alpha_\gamma(a^* b(s(\gamma)))d\lambda^u(\gamma) = c\langle a, b\rangle_D(u)$$

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Theorem (B 2008)

Let G be a groupoid acting properly on a C^*-algebra A (via α) with respect to the dense A_0. Then the generalized fixed point algebra A^α for this action is Morita Equivalent to a subalgebra of the reduced crossed product.

Examples

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  - In this case $A^\alpha = C_0(G\backslash X)$
- (B 2008) If G is a proper groupoid acting on A
  - A must necessarily be a $C_0(G^{(0)})$- algebra
  - G acts properly on A with respect to the dense subalgebra $C_c(G^{(0)}) \cdot A$.

Future Directions

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- Saturated
  - Rieffel calls a proper action saturated if A^α Morita Equivalent to the reduced crossed product
  - I would like to find some conditions that guarantee that an action is saturated.