A noncommutative extended de Finetti theorem

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Plan

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Motivation

> *Though many probabilistic symmetries are conceivable [...], four of them -* **stationarity**, **contractability**, **exchangeablity** *and* **rotatability** *- stand out as especially interesting and important in several ways: Their study leads to some deep structural theorems of great beauty and significance [...].*
>
> *Olav Kallenberg (2005)*

Question:

Can one transfer the related concepts to **noncommutative probability theory** and do they turn out to be fruitful in the study of the structure of **operator algebras**?

Hierarchy of distributional symmetries

invariant objects	transformations
stationary	shifts
contractable	sub-sequences
exchangeable	permutations
rotatable	isometries

Topic of this talk:

- invariant objects are generated by an **infinite sequence** of random variables
- only the **first three symmetries** are considered
- **contractable = spreadable**

Motivating Example for De Finetti theorem

"Any exchangeable process is an average of i.i.d. processes."

Theorem (De Finetti 1931)

$X_1, X_2, \ldots$ infinite sequence of $\{0,1\}$-valued random variables s.t.

$$P(X_1 = e_1, \ldots, X_n = e_n) = P(X_{\pi(1)} = e_1, \ldots, X_{\pi(n)} = e_n)$$

holds for all $n \in \mathbb{N}$, permutations π and every $e_1, \ldots, e_n \in \{0,1\}$.
Then there exists a unique probability measure μ on $[0,1]$ such that

$$P(X_1 = e_1, \ldots, X_n = e_n) = \int p^s(1-p)^{n-s} d\mu(p),$$

where $s = e_1 + e_2 + \ldots + e_n$.

Terminology of noncommutative probability

- **Probability space** $(\mathcal{A}, \varphi)$
  $\mathcal{A} =$ von Neumann algebra with separable predual
  $\varphi =$ faithful normal state on $\mathcal{A}$
- **Random variable** $\iota \colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{A}, \varphi)$
  $\iota \colon \mathcal{A}_0 \to \mathcal{A}$ injective *-homomorphism satisfying

$$\iota(\mathbb{1}_{\mathcal{A}_0}) = \mathbb{1}_{\mathcal{A}}, \qquad \varphi \circ \iota = \varphi_0 \qquad \text{and} \qquad \iota\, \sigma_t^{\varphi_0} = \sigma_t^{\varphi}\, \iota$$

- **Automorphisms of** $(\mathcal{A}, \varphi)$
  $\mathrm{Aut}(\mathcal{A}, \varphi) =$ *-automorphisms α of $\mathcal{A}$ with $\varphi \circ \alpha = \varphi$

Remark

For commutative algebra of functions we get classical random variables and measure preserving transformations.

Noncommutative independence and commuting squares

Given the probability space $(\mathcal{A}, \varphi)$, let $\mathcal{A}_0 \subset \mathcal{A}_1, \mathcal{A}_2$ be three von Neumann subalgebras of $\mathcal{A}$ such that the φ-preserving conditional expectations $E_i \colon \mathcal{A} \to \mathcal{A}_i$ exist $(i = 1, 2, 3)$. Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are said to be $\mathcal{A}_0$-**independent** if one of the following (equivalent) conditions is satisfied:

1. $E_0(xy) = E_0(x)E_0(y)$ for all $x \in \mathcal{A}_1$ and $y \in \mathcal{A}_2$
2. $E_1 \circ E_2 = E_0$
3. the diagram $$\begin{tikzcd} \mathcal{A}_1 \arrow[r, phantom, "\subset"] & \mathcal{A} \\ \mathcal{A}_0 \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\subset"] & \mathcal{A}_2 \arrow[u, phantom, "\cup"] \end{tikzcd}$$ is a **commuting square**

Remark

For commutative algebras of functions on a probability space we obtain conditional independence with respect to a sub-σ-algebra.

Conditional Independence of random sequences

Let $I, J \subset \mathbb{N}_0$. A sequence of random variables, for short: *random sequence*,

$$\iota \equiv (\iota_n)_{n\geq 0} \colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{A}, \varphi)$$

is $\mathcal{B}$-**independent** if

$$\bigvee\{\iota_i(\mathcal{A}_0) \,|\, i \in I\} \vee \mathcal{B} \quad \text{and} \quad \bigvee\{\iota_j(\mathcal{A}_0) \,|\, j \in J\} \vee \mathcal{B}$$

are $\mathcal{B}$-independent whenever $I \cap J = \emptyset$.

Remark

$\mathcal{B}$ may **not** be contained in $\bigvee\{\iota_i(\mathcal{A}_0) \,|\, i \in I\}$

Noncommutative distributions

Two random sequences $\iota \equiv (\iota_n)_{n\geq 0}$ and $\tilde{\iota} \equiv (\tilde{\iota}_n)_{n\geq 0}$ from $(\mathcal{A}_0, \varphi_0)$ to $(\mathcal{A}, \varphi)$ have the same **noncommutative distribution** if

$$\begin{array}{c}\varphi\big(\iota_{\mathbf{i}(1)}(a_1)\,\iota_{\mathbf{i}(2)}(a_2)\cdots\iota_{\mathbf{i}(n)}(a_n)\big)\\ \Vert \\ \varphi\big(\tilde{\iota}_{\mathbf{i}(1)}(a_1)\,\tilde{\iota}_{\mathbf{i}(2)}(a_2)\cdots\tilde{\iota}_{\mathbf{i}(n)}(a_n)\big)\end{array}$$

for all n-tuples $\mathbf{i}\colon \{1, 2, \ldots, n\} \to \mathbb{N}_0$, $(a_1, \ldots, a_n) \in \mathcal{A}_0^n$ and $n \in \mathbb{N}$.

Notation

$$(\iota_0, \iota_1, \iota_2, \ldots) \overset{\text{distr}}{=} (\tilde{\iota}_0, \tilde{\iota}_1, \tilde{\iota}_2, \ldots)$$

Noncommutative distributional symmetries

Just as in the classical case we can now talk about distributional symmetries. A random sequence $\iota \equiv (\iota_n)_{n\geq 0}$ is

- **exchangeable** if $(\iota_0, \iota_1, \iota_2, \ldots) \overset{\text{distr}}{=} (\iota_{\pi(0)}, \iota_{\pi(1)}, \iota_{\pi(2)}, \ldots)$ for any finite permutation $\pi \in \mathbb{S}_\infty$ of $\mathbb{N}_0$.
- **spreadable** if $(\iota_0, \iota_1, \iota_2, \ldots) \overset{\text{distr}}{=} (\iota_{n_0}, \iota_{n_1}, \iota_{n_2}, \ldots)$ for any subsequence $(n_0, n_1, n_2, \ldots)$ of $(0, 1, 2, \ldots)$.
- **stationary** if $(\iota_0, \iota_1, \iota_2, \ldots) \overset{\text{distr}}{=} (\iota_k, \iota_{k+1}, \iota_{k+2}, \ldots)$ for all $k \in \mathbb{N}$.

Lemma (Hierarchy of distributional symmetries)

Exchangeability $\Rightarrow$ Spreadability $\Rightarrow$ Stationarity

Classical dual version of extended De Finetti theorem

Let $\iota \equiv (\iota_n)_{n\geq 0}\colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{A}, \varphi)$ be a random sequence with

$$\mathcal{A}^{\text{tail}} := \bigcap_{n\geq 0} \bigvee_{k\geq n} \iota_k(\mathcal{A}_0)$$

and consider:

(a) ι is exchangeable

(c) ι is spreadable

(e) ι is $\mathcal{A}^{\text{tail}}$-independent

Theorem (De Finetti '31, Ryll-Nardzewski '57,..., K. '08)

$\mathcal{A} \simeq L^\infty(\mathrm{A}, \Sigma, \mu)$ *implies:* (a) $\Leftrightarrow$ (c) $\Leftrightarrow$ (e)

Remark

Idea of dual version seems to be new!

Noncommutative De Finetti theorems with commutativity conditions

Let $\iota \equiv (\iota_n)_{n\geq 0}\colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{A}, \varphi)$ be a random sequence with

$$\mathcal{A}^{\mathrm{tail}} := \bigcap_{n\geq 0} \bigvee_{k\geq n} \iota_k(\mathcal{A}_0)$$

and consider:

(a) ι is exchangeable

(e) ι is $\mathcal{A}^{\mathrm{tail}}$-independent

Theorem (Størmer '69, ..., K. '08)

Ranges of ι_n's commute: $\quad$ *(a) $\Leftrightarrow$ (e)*

Theorem (Accardi & Lu '93, ..., K. '08)

$\mathcal{A}^{\mathrm{tail}}$ *is abelian:* $\quad$ *(a) $\Rightarrow$ (e)*

Noncommutative extended De Finetti theorem

Let $\iota \equiv (\iota_n)_{n\geq 0}\colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{A}, \varphi)$ be a random sequence with

$$\mathcal{A}^{\text{tail}} := \bigcap_{n\geq 0} \bigvee_{k\geq n} \iota_k(\mathcal{A}_0)$$

and consider:

(a) ι is exchangeable

(c) ι is spreadable

(d) ι is stationary and $\mathcal{A}^{\text{tail}}$-independent

(e) ι is $\mathcal{A}^{\text{tail}}$-independent

Theorem (K. '07-'08)

(a) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e), but (a) $\not\Leftarrow$ (c) $\not\Leftarrow$ (d) $\not\Leftarrow$ (e)

Remark

'Spreadability' implies 'conditional independence' is the hard part!

Artin braid groups $\mathbb{B}_n$

Algebraic Definition (Artin 1925)

$\mathbb{B}_n$ is presented by $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ satisfying

$$\begin{aligned} \sigma_i\sigma_j\sigma_i &= \sigma_j\sigma_i\sigma_j && \text{if } \mid i-j \mid = 1 && \text{(B1)} \\ \sigma_i\sigma_j &= \sigma_j\sigma_i && \text{if } \mid i-j \mid > 1 && \text{(B2)} \end{aligned}$$

Figure: Artin generators σ_i (left) and σ_i^{-1} (right)

$\mathbb{B}_1 \subset \mathbb{B}_2 \subset \mathbb{B}_3 \subset \ldots \subset \mathbb{B}_\infty$ (inductive limit)

Braidability

Definition (Gohm & K. '08)

A random sequence $\iota \equiv (\iota_n)_{n\geq 0}\colon (\mathcal{A}_0, \varphi_0) \to (\mathcal{A}, \varphi)$ is **braidable** if there exists a representation $\rho\colon \mathbb{B}_\infty \to \mathrm{Aut}(\mathcal{A}, \varphi)$ satisfying:

$$\begin{aligned} \iota_n &= \rho(\sigma_n \sigma_{n-1} \cdots \sigma_1)\iota_0 && \text{for all } n \geq 1; \\ \iota_0 &= \rho(\sigma_n)\iota_0 && \text{if } n \geq 2. \end{aligned}$$

Braidability extends exchangeability

- If $\rho(\sigma_n^2) = \mathrm{id}$ for all n, one has a representation of $\mathbb{S}_\infty$.
- ι is exchangeable $\Leftrightarrow$ ι is braidable with $\rho(\sigma_n^2) = \mathrm{id}$ for all n.

Braidability implies spreadability

It turns out that we can insert braidability between exchangeability and spreadability in the noncommutative de Finetti theorem and obtain a large and interesting class of spreadable sequences in this way.

Consider the conditions:

(a) ι is exchangeable
(b) ι is **braidable**
(c) ι is spreadable
(d) ι is stationary and $\mathcal{A}^{\text{tail}}$-independent

Theorem (Gohm & K. '08)

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d), \quad \text{but } (a) \not\Leftarrow (b) \not\Leftarrow (d)$$

Open Problem

Construct spreadable sequence which fails to be braidable.

Wang's quantum permutation group (1998)

The **quantum permutation group** $A_s(n)$ is the universal unital C*-algebra generated by elements $\{u_{ij} \mid i,j = 1,\dots n\}$ such that

1. $u_{ij}^* = u_{ij} = u_{ij}^2$ (orthogonal projections)
2. the elements of each row and each column of the matrix

$$\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots \\ u_{n1} & \cdots & & u_{nn} \end{bmatrix}$$

form a partition of unity, i.e. are orthogonal and sum up to 1.

Remark

$A_s(n)$ is a compact quantum group in the sense of Woronowicz

Quantum Exchangeability

Given a probability space $(\mathcal{A}, \varphi)$ with an infinite sequence $x_1, x_2, \ldots \subset \mathcal{A}$, define the action of $A_s(k)$ on the tuple $(x_1, x_2, \ldots, x_k)$ by

$$x_i \mapsto \sum_{i=1}^{k} u_{ij} \otimes x_j \in A_s(k) \otimes \mathcal{A}.$$

Definition (K. & Speicher '08)

The distribution of $(x_1, x_2, \ldots)$ is **quantum exchangeable** if

$$\varphi\left(x_{\mathbf{i}(1)} \cdots x_{\mathbf{i}(n)}\right) = \sum_{\mathbf{j}(1),\ldots,\mathbf{j}(n)=1}^{k} u_{\mathbf{i}(1)\mathbf{j}(1)} \cdots u_{\mathbf{i}(n)\mathbf{j}(n)} \cdot \varphi\left(x_{\mathbf{j}(1)} \cdots x_{\mathbf{j}(n)}\right)$$

for all $1 \leq \mathbf{i}(1), \ldots \mathbf{i}(n) \leq k$ and all $k, n \in \mathbb{N}$.

A new characterization of freeness with amalgamation

Theorem (K. & Speicher '08)

Let $(\mathcal{A}, \varphi)$ be a probability space and consider an infinite sequence $x_1, x_2, \ldots \in \mathcal{A}$. Then the following two statements are equivalent:

(a) $(x_n)_{n\in\mathbb{N}}$ is quantum exchangeable

(b) $(x_n)_{n\in\mathbb{N}}$ is identically distributed and free over $\mathcal{A}^{\mathrm{tail}}$

Corollary (K. & Speicher '08

Suppose $(x_n)_{n\in\mathbb{N}}$ is quantum exchangeable. If $\mathcal{A}^{\mathrm{tail}}$ is trivial, then $(x_n)_{n\in\mathbb{N}}$ is free.

References

- *A noncommutative extended de Finetti theorem.* `http://arxiv.org/pdf/0806.3621.pdf`
- *Noncommutative independence from the braid group* $\mathbb{B}_\infty$. (With Rolf Gohm) `http://arxiv.org/pdf/0806.3691.pdf`
- *On Lehner's 'free' noncommutative analogue of the de Finetti theorem.* `http://arxiv.org/pdf/0806.3632.pdf`
- *A noncommutative de Finetti theorem: Invariance under quantum permutations is equivalent to freeness with amalgamation.* (With Roland Speicher) `Preprint 2008, to be posted at arxiv.org`