On *k*-morphs.

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What are k-morphs?

Many interesting C*-algebras arise from the study of k-graphs.
Morphs provide a way of combining k-graphs.

- A k-morph between two k-graphs yields a $(k+1)$-graph.
- Isomorphism classes of k-morphs form a category $\mathcal{M}_k$.
- Let $\mathcal{C}^*$ be the category of C*-correspondences.
  Then there is a functor $\mathcal{M}_k^{\maltese} \to \mathcal{C}^*$.

k-graphs.

Definition: Let Λ be a countable small category and let $d : \Lambda \to \mathbb{N}^k$ be a functor. We say (Λ, d) is a *k-graph*, if for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that

$$d(\lambda) = m + n$$

there exist unique $\mu, \nu \in \Lambda$ satisfying:

- $d(\mu) = m$ and $d(\nu) = n$,
- $\lambda = \mu\nu$.

Notation: Set $\Lambda^n := d^{-1}(n)$ and identify $\Lambda^0 = \mathrm{Obj}\,(\Lambda)$.

Remarks.

Let Λ be a k-graph.

- If $k = 0$, then d is trivial and Λ is just a set.
- If $k = 1$, then Λ is the path category of a directed graph.
- If $k \geq 2$, think of Λ as generated by k graphs of different colors that share the same set of vertices Λ^0.

Commuting squares, cubes etc. form an essential piece of structure in the last case.

The C*-algebra $C^*(\Lambda)$.

Suppose $(*)$ $\forall v \in \Lambda^0, n \in \mathbb{N}^k$, $v\Lambda^n$ is finite and nonempty where

$$v\Lambda^n := r^{-1}(v) \cap \Lambda^n.$$

Definition: Let $C^*(\Lambda)$ be the universal C*-algebra generated by the set $\{s_\lambda : \lambda \in \Lambda\}$ satisfying:

- for $s(\lambda) = v$, ${s_\lambda}^* s_\lambda = s_v$,
- for $v \in \Lambda^0, n \in \mathbb{N}^k$,

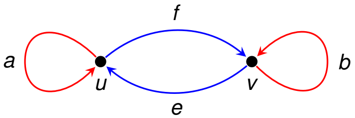
$$s_v = \sum_{\lambda \in v\Lambda^n} s_\lambda {s_\lambda}^*,$$

- for $\lambda, \mu \in \Lambda$,

$$s_\lambda s_\mu = \begin{cases} s_{\lambda\mu} & \text{if } s(\lambda) = r(\mu), \\ 0 & \text{else.} \end{cases}$$

Example of a 2-graph.

The following is a representation of a 2-graph Λ. Only the morphisms of minimal degree, $\Lambda^{(1,0)}$ and $\Lambda^{(0,1)}$, are presented. The commutation rules are uniquely determined.

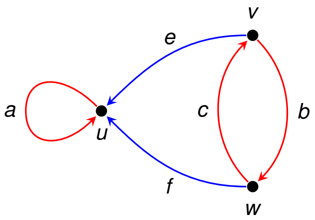


We have

$$C^*(\Lambda) \cong M_2 \otimes C(\mathbb{T}^2).$$

Another Example of a 2-graph.

Here is another 2-graph Σ. Only $\Sigma^{(1,0)}$ and $\Sigma^{(0,1)}$ are given.



We have

$$C^*(\Sigma) \cong M_4 \otimes C(\mathbb{T}).$$

Definition.

Let Λ and Γ be k-graphs. Let X be a countable set and let $r : X \to \Lambda^0$, $s : X \to \Gamma^0$ and $\varphi : X * \Gamma \to \Lambda * X$ be given. Suppose:

- φ is a bijection,
- if $\varphi(x_1, \gamma_1) = (\lambda_1, x_2)$, the following holds:
  - $d(\gamma_1) = d(\lambda_1)$, $r(x_1) = r(\lambda_1)$ and $s(\gamma_1) = s(x_2)$,
  - If $\varphi(x_2, \gamma_2) = (\lambda_2, x_3)$, then $\varphi(x_1, \gamma_1\gamma_2) = (\lambda_1\lambda_2, x_3)$.

Then the quadruple (X, r, s, φ) is said to be a Λ-Γ *morph*.

Often, we just say that X is a k-morph.

If $\Lambda = \Gamma$, say X is a Λ-*endomorph*.

Linking Graph.

For a Λ-Γ morph X, there is a $(k+1)$-graph Σ s.t.

- $\Sigma^{(n,0)} = \Lambda^n \sqcup \Gamma^n$ for $n \in \mathbb{N}^k$,
- $\Sigma^{(0,1)} = X$,
- under these identifications,

$$x_1\gamma = \lambda x_2 \quad \text{iff} \quad \varphi(x_1,\gamma) = (\lambda, x_2).$$

Every k-morph arises in this way. The second example above is a linking graph for the 1-morph $X = \{e, f\}$.

The C*-correspondence $\mathcal{H}(X)$.

Suppose Σ satisfies $(*)$; then $C^*(\Sigma)$ makes sense. Set

$$P := \sum_{u \in \Lambda^0} s_u \quad \text{and} \quad Q := \sum_{v \in \Lambda^0} s_v.$$

Proposition: With notation as above $\mathcal{H}(X) := PC^*(\Sigma)Q$ may be viewed as a $C^*(\Lambda)$-$C^*(\Gamma)$ correspondence.

The left action induces an embedding $C^*(\Lambda) \hookrightarrow \mathcal{K}(\mathcal{H}(X))$.

Composing Morphs.

Given a Λ-Γ morph X and a Γ-Σ morph Y,
we construct a Λ-Γ morph

$$Z = X * Y = \{(x, y) : s(x) = r(y)\},$$

by defining

$$\varphi_Z(x, y, \sigma) = (\lambda, x', y'),$$

if there is $\gamma \in \Gamma$ such that

$$\varphi_X(x, \gamma) = (\lambda, x'),$$
$$\varphi_Y(y, \sigma) = (\gamma, y').$$

The Category $\mathcal{M}_k$.

This composition allows one to define the category $\mathcal{M}_k$ as follows:

- $\mathrm{Obj}\,(\mathcal{M}_k)$ consists of k graphs,
- $\mathrm{Mor}_{\mathcal{M}_k}(\Gamma, \Lambda) =$ iso. classes of Λ-Γ morphs,
- composition is given by

$$[X] \circ [Y] = [X * Y].$$

The Subcategory $\mathcal{M}_k^{\maltese}$.

The subcategory $\mathcal{M}_k^{\maltese}$ consists of k-graphs satisfying ($*$) and k-morphs satisfying a mild condition.

Lemma: Given a Λ-Γ morph X and a Γ-Σ morph Y in $\mathcal{M}_k^{\maltese}$:

$$\mathcal{H}(X * Y) \cong \mathcal{H}(X) \otimes_{C^*(\Gamma)} \mathcal{H}(Y).$$

Let $\mathcal{C}^*$ be the category of C*-algebras and iso. classes of C*-correspondences.

Theorem: There is a functor from $\mathcal{M}_k^{\maltese}$ to $\mathcal{C}^*$ given by

$$\Lambda \mapsto C^*(\Lambda),$$
$$[X] \mapsto [\mathcal{H}(X)].$$

Endomorphs.

For a Λ-endomorph X, there is a $(k+1)$-graph $\Sigma = \Lambda \times_X \mathbb{N}$ s.t.

- $\Sigma^{(n,0)} = \Lambda^n$ and $\Sigma^{(0,1)} = X$,
- under these identifications,

$$x_1\lambda_1 = \lambda_2 x_2 \quad \text{iff} \quad \varphi(x_1, \lambda_1) = (\lambda_2, x_2).$$

Then $\Lambda \times_X \mathbb{N}$ is called the *endomorph skew-graph*.

Theorem: For X and Λ in $\mathcal{M}_k^{\maltese}$ we have:

$$C^*(\Lambda \times_X \mathbb{N}) \cong \mathcal{O}_{\mathcal{H}(X)}.$$