

On Higher Real and Stable Ranks for CCR C^* -algebras

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Stable rank: $1 \leq \text{tsr}(A) \leq \infty$.

Real rank: $0 \leq \text{RR}(A) \leq \infty$.

$\text{tsr}(C_0(X)) = [\dim(X \cup \{\infty\})/2] + 1$, and

$\text{RR}(C_0(X)) = \dim(X \cup \{\infty\})$.

Rieffel (1983): $\text{tsr}(A)$ is the smallest n such that unimodular n -tuples are dense in \tilde{A}^n

B-Pedersen (1991): $\text{RR}(A)$ is the smallest n such that unimodular self-adjoint n -tuples are dense in $\tilde{A}_{\text{sa}}^{n+1}$.

A unimodular n -tuple is one that is left invertible as an $n \times 1$ matrix.

(1) $\text{tsr}(A) = \max(\text{tsr}(I), \text{tsr}(A/I))$, or

(2) $\text{tsr}(A) = 1 + \max(\text{tsr}(I), \text{tsr}(A/I))$.

Questions. Does one always have

$\text{tsr}(A) \leq \max(\text{tsr}(I), \text{tsr}(A/I), 2)$?

$\text{RR}(A) \leq \max(\text{RR}(I), \text{RR}(A/I), 1)$?

Lacking positive answers to these questions, it is worthwhile to find examples of special ideals I for which the answer is yes. Nistor did this in 1987 when I arises from a locally trivial bundle of elementary C^* -algebras over a finite dimensional space.

The low ranks and higher ranks have different formal properties. For example, the low ranks are invariant under Rieffel–Morita equivalence, whereas $\text{tsr}(A \otimes \mathbb{K}) = 2$ whenever $\text{tsr}(A) > 1$ and $\text{RR}(A \otimes \mathbb{K}) = 1$ whenever $\text{RR}(A) > 0$.

Another very general difference between low ranks and higher ranks, which plays a crucial role in our positive results on extensions, will appear later.

In general much less is known about higher real rank than higher stable rank. For example, it was only in 1995 that it was proved, by Hassan, that $\text{RR}(I) \leq \text{RR}(A)$ when I is an ideal of A . Nevertheless, for the algebras treated in this work, I am able to treat real rank and stable rank in parallel ways.

Some results from my second-to-last joint paper with Gert Pedersen (Limits and C^* -algebras of low rank or dimension, J. Operator Theory, in press, arxiv no. 0708.2727) play a key role in this work.

One is a notion of topological dimension for C^* -algebras with almost Hausdorff primitive ideal space. This in particular applies to all type I C^* -algebras. Although $\text{top dim}(A)$ is a property of the primitive ideal space of A , and its theory isn't a big deal, it is important to use the right notion and not just apply covering dimension to the generally non-Hausdorff spaces which occur. Even when $\text{prim}(A)$ is Hausdorff, we need to define $\text{top dim}(A)$ as the local dimension of $\text{prim}(A)$, which may differ from the global dimension if A is not σ -unital.

$\text{top dim}(A) = \sup_{K \subset \text{prim}(A)} \{\dim(K)\}$, where K is compact Hausdorff.

The second is a result about rank which is analogous to and inspired by the countable sum theorem of topological dimension theory. It is also partly inspired by a 1987 paper of Sheu. Parts (i) and (ii) of Theorem 2.12 of the cited B-Pedersen paper can be stated as follows:

(CST) If $\text{prim}(A) = \bigcup_{n=1}^{\infty} F_n$, where each F_n is closed, then

$\text{tsr}(A) = \sup_n \{\text{tsr}(A/I_n)\}$ and

$\text{RR}(A) = \sup_n \{\text{RR}(A/I_n)\}$.

Here I_n is the kernel of F_n , so that $\text{prim}(A/I_n) = F_n$.

Some more background:

If A is commutative, then by Rieffel's results,

$$\text{tsr}(A \otimes \mathbb{M}_n) = \lceil \frac{2n-1+d}{2n} \rceil = \lfloor \frac{4n-2+d}{2n} \rfloor;$$

and by Beggs-Evans (1991),

$$\text{RR}(A \otimes \mathbb{M}_n) = \lceil \frac{d}{2n-1} \rceil = \lfloor \frac{d+2n-2}{2n-1} \rfloor.$$

Here $d = \text{top dim}(A)$ ($d = 0$ if $A = \{0\}$).

(CST) makes it easy to extend these results to n -homogeneous C^* -algebras.

Theorem 3.9. *Let A be a C^* -algebra with only finite dimensional irreducible representations and $H_n = \{P \in \text{prim}(A) : A/P \cong \mathbb{M}_n\}$. Then if $\text{top dim}(A(H_n)) (= \text{loc dim}(H_n)) = d_n$, we have*

(i) $\text{tsr}(A) = \sup_n \{\lceil \frac{2n-1+d_n}{2n} \rceil\}$, and

(ii) $\text{RR}(A) = \sup_n \{\lceil \frac{d_n}{2n-1} \rceil\}$.

Theorem 3.10. *Let A be a CCR C^* -algebra, and suppose that $d = \text{top dim}(A) < \infty$. Let $H_n = \{P \in \text{prim}(A) : A/P \cong \mathbb{M}_n\}$, and let $d_n = \text{top dim}(A(H_n))$.*

(i) *If $d \leq 1$, then $\text{tsr}(A) = 1$.*

(ii) *If $d > 1$, then $\text{tsr}(A) = \sup_n \{\max(\lceil \frac{2n-1+d_n}{2n} \rceil, 2)\}$.*

(iii) *If $d = 0$, then $\text{RR}(A) = 0$.*

(iv) *If $d > 0$, then $\text{RR}(A) = \sup_n \{\max(\lceil \frac{d_n}{2n-1} \rceil, 1)\}$.*

Here $\text{prim}(A(H_n)) = H_n$.

Theorem 3.6. *If A is a CCR C^* -algebra and I a closed two-sided ideal, then $\text{rank}(A) = \max(\text{rank}(I), \text{rank}(A/I))$.*

Nistor used his result on extensions as the main tool in calculating stable ranks of certain C^* -algebras, and my 3.6 is used in proving the two previously stated theorems. However, the hypothesis of 3.6 is very restrictive. The next three extension results are conscious generalizations of Nistor's result.

Theorem 3.11. *If I is an ideal of the C^* -algebra A such that all irreducible representations of I are finite dimensional, then $\text{rank}(A) = \max(\text{rank}(I), \text{rank}(A/I))$.*

The concept of generalized continuous trace (GCT) was defined by Dixmier (1963). Let $J(A)$ denote the closure of the set of continuous trace elements of A . Then $J(A)$ is the largest ideal of A such that $J(A)$ has continuous trace as a C^* -algebra and every compact subset of $\text{prim}(J(A))$ is closed in $\text{prim}(A)$. The continuous trace composition series is $\{J_\alpha : 0 \leq \alpha \leq \beta\}$, where β is an ordinal number, $J_0 = 0$, $J_\lambda = (\cup_{\alpha < \lambda} J_\alpha)^-$ for λ a limit ordinal, $J_{\alpha+1}/J_\alpha = J(A/J_\alpha) \neq 0$ for $\alpha < \beta$, and $J(A/J_\beta) = 0$. Then A is GCT if and only if $J_\beta = A$. Although every type I C^* -algebra has a composition series with continuous trace quotients, every GCT C^* -algebra is CCR . Dixmier proved that GCT algebras are distinguished from other type I C^* -algebras by the topology of their spectra.

Theorem 3.12. *Let I be an ideal of the C^* -algebra A such that I is separable, I has generalized continuous trace, $\text{top dim}(I) < \infty$, and all irreducible representations of I are infinite dimensional. Then*

(i) $\text{tsr}(A) \leq \max(2, \text{tsr}(A/I))$, and

(ii) $\text{RR}(A) = \max(\text{RR}(I), \text{RR}(A/I))$.

Corollary 3.15. *Assume that I is a separable stable ideal of the C^* -algebra A and that I has the corona factorization property and has an approximate identity consisting of projections. Then*

(i) $\text{tsr}(A) \leq \max(\text{tsr}(A/I), 2)$, and

(ii) $\text{RR}(A) \leq \max(\text{RR}(A/I), 1)$.

One of the main ideas in the proofs of the last two results arises from a simple consideration which illustrates another difference between the low rank and higher rank cases. Suppose \underline{x} is a n -tuple, for $n > 1$, which arises in checking the stable or real rank of A . Is it possible that \underline{x} is actually right invertible?

For the stable rank case it follows from Rieffel's results that the answer is no. If \underline{x} is right invertible, then A is properly infinite and this implies that $\text{tsr}(A) = \infty$.

For the real rank case the answer is also no.

Lemma 2.6. *Let A be a non-zero unital C^* -algebra, and let \underline{h} be an n -tuple in $(A_{sa})^n$, where $n \geq 2$. Then the $n \times n$ matrix $(h_i h_j)$ is not invertible.*

Proof. Regard \underline{h} as an $n \times 1$ matrix, so that the matrix in question is $\underline{h} \underline{h}^*$. If A is a unital subalgebra of $B(\mathcal{H})$, then \underline{h} may be regarded as an operator from \mathcal{H} to $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$. Then if $\underline{h} \underline{h}^*$ is invertible, \underline{h} must be surjective. It follows that each h_i is surjective and (since $n > 1$) no h_i is injective. This is absurd, since h_i is self-adjoint.

I learned about generalized continuous trace algebras from Phil Green in 1976, and he proved a very nice new characterization in a letter to me later that year. I failed to persuade him to publish this result, but he has given me permission to include it in the paper. Although GCT is mentioned in only one of my results, and Green's result is not used, you will see that the GCT concept is hovering in the background; and in fact Green's result helped me develop the perspective needed for my work.

Definition 3.2. If X is a primitive ideal space, then an *FD-like decomposition* of X is a family $\{H_1, H_2, \dots\}$ of locally closed subsets of X such that:

(i) $X = \bigcup_n H_n$, $H_n \cap H_m = \emptyset$ if $n \neq m$.

(ii) Each H_n is Hausdorff.

(iii) Every compact subset of H_n is closed in X .

(iv) $F_n = \bigcup_{k=1}^n H_k$ is closed.

Lemma 3.4. *If $\{H_n\}$ is an FD-like decomposition of $\text{prim}(A)$, then $\text{tsr}(A) = \sup_n \{\text{tsr}(A(H_n))\}$ and $\text{RR}(A) = \sup_n \{\text{RR}(A(H_n))\}$.*

Theorem 4.1 (Green). *If A is a separable CCR C^* -algebra, then the following are equivalent:*

(i) *A has generalized continuous trace.*

(ii) *Every non-empty closed subset F of $\text{prim}(A)$ has a non-empty relatively open subset G such that G is Hausdorff and every compact subset of G is closed in $\text{prim}(A)$.*

(iii) *Every non-empty closed subset F of $\text{prim}(A)$ has a non-empty relatively open subset G such that if $x \in G, y \in F$, and $x \neq y$, then x and y have disjoint neighborhoods relative to F .*

(iv) *Every non-empty closed subset F of $\text{prim}(A)$ has a non-empty relatively open subset G such that each point of G has a (relative) neighborhood base consisting of sets closed in $\text{prim}(A)$.*

(v) *One can write $\text{prim}(A) = \bigcup_1^\infty F_n$, where $\{F_n\}$ is a countable family of closed compact sets.*

(vi) *The space $\text{prim}(A)$ is metacompact; i.e., every open cover has an open point–finite refinement.*

(vii) *There is a point–finite open cover $\{U_i\}$ of $\text{prim}(A)$ such that each U_i is contained in a compact subset of $\text{prim}(A)$.*

(viii) *A is stably isomorphic to a C^* –algebra with only finite dimensional irreducible representations.*

Corollary 4.2. *If A is a separable CCR C^* –algebra, then A has generalized continuous trace if and only if $\text{prim}(A)$ has an FD–like decomposition.*