Noncommutative Khintchine-type inequalities and applications

(Joint work with Uffe Haagerup)

Magdalena Musat
University of Memphis

GPOTS 2008
University of Cincinnati

June 18, 2008

Khintchine inequalities: Let $r_n(t) := \mathrm{sgn}(\sin(2^n t\pi))$, $n \geq 1$ be the Rademacher functions on $[0,1]$.
Then for every $0 < p < \infty$, there exist A_p , $B_p > 0$ such that for arbitrary $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{R}$,

$$A_p \left(\sum_{k=1}^{n} a_k^2 \right)^{1/2} \leq \left(\int_0^1 \left| \sum_{k=1}^{n} a_k r_k \right|^p dt \right)^{1/p} \leq B_p \left(\sum_{k=1}^{n} a_k^2 \right)^{1/2}$$

Suppose that A_p , B_p already denote the best constants above. By elementary methods, $B_p = 1$, $0 < p \leq 2$ and $A_p = 1$, $2 \leq p < \infty$. Furthermore,

- Szarek (1974): $A_1 = 1/\sqrt{2}$.
- Young (1976): computed B_p for $p \geq 3$.
- Haagerup (1982): computed A_p and B_p in the remaining cases.

Kahane proved that for any Banach space X,

$$A_{p,X} \left\| \sum a_k r_k \right\|_{L^2([0,1];X)} \leq \left\| \sum a_k r_k \right\|_{L^p([0,1];X)} \leq B_{p,X} \left\| \sum a_k r_k \right\|_{L^2([0,1];X)}$$

In particular, if $X = L^p(\Omega, \mu)$, $1 \leq p < \infty$, then

$$A'_p \left\| \left(\sum |a_k|^2 \right)^{1/2} \right\|_{L^p(\Omega)} \leq \left\| \sum a_k r_k \right\|_{L^p([0,1];L^p(\Omega))} \leq B'_p \left\| \left(\sum |a_k|^2 \right)^{1/2} \right\|_{L^p(\Omega)}$$

Next step: Generalize this to the case of noncommutative L^p-spaces, e.g., the Schatten classes S_p. Recall that for $n \geq 1$,

$$\|x\|_{S_p^n} := \Big(\mathrm{Tr}((x^*x)^{p/2})\Big)^{1/p}, \quad x \in M_n(\mathbb{C}),$$

were Tr denotes the non-normalized trace on $M_n(\mathbb{C})$.
These results were obtained by Lust-Piquard (1986) for $1 < p < \infty$, and by Lust-Piquard and Pisier (1991) for $p = 1$.

Theorem (**Lust-Piquard-Pisier 1991**):
Given $d, n \in \mathbb{N}$ and $x_1, \ldots, x_d \in M_n(\mathbb{C})$, then

$$\frac{1}{1+\sqrt{2}}\, |||\{x_j\}_{j=1}^d|||^* \leq \left\| \sum_{j=1}^d x_j \otimes e^{i2^n t} \right\|_{L^1([0,1];S_1^n)} \leq |||\{x_j\}_{j=1}^d|||^* \qquad (1)$$

where, by definition,

$$|||\{x_i\}_{i=1}^d|||^* := \inf \left\{ \left\| \sum_{i=1}^d y_i^* y_i \right\|_{S_1^n} + \left\| \sum_{i=1}^d z_i z_i^* \right\|_{S_1^n} ; x_i = y_i + z_i \right\}.$$

Remark: By classical results of Maurey and Pisier, inequalities (1) will then also hold when replacing the sequence $\{e^{i2^n t}\}_{n\geq 1}$ by a sequence of Rademacher functions, or independent standard complex Gaussian random variables, or Steinhauss random variables, but with possibly different constants.

In joint work with U. Haagerup (JFA 250 (2007)) we obtain a direct proof of inequalities (1) both in the Gaussian and the Rademacher case.

The Gaussian case:

Theorem 1 (**Haagerup-M. 2007**)

Let $d\,, n \geq 1$ and consider $x_1, \ldots, x_d \in M_n(\mathbb{C})$. Let $\{\gamma_i\}_{1\leq i\leq n}$ be independent, standard complex-valued Gaussian random variables on a probability space $(\Omega, \mathbb{P})$. Then

$$\frac{1}{\sqrt{2}}\,|||\{x_i\}_{i=1}^d|||^* \;\leq\; \left\|\sum_{i=1}^d x_i\otimes\gamma_i\right\|_{L^1(\Omega;S_1^n)} \leq |||\{x_i\}_{i=1}^d|||^*\,.$$

Remark: The conclusion of Theorem 1 remains valid when replacing $\{\gamma_n\}_{n\geq 1}$ by $\{e^{i2^n t}\}_{n\geq 1}$, or by a sequence of Steinhauss r. v. Moreover, we show that in all these cases, both lower and upper bound are sharp.

Theorem 2 (**Haagerup-M. 2007**)

Denote by $c_1\,, c_2$ the best constants in the inequalities

$$c_1|||\{x_i\}_{i=1}^d|||^* \;\leq\; \left\|\sum_{i=1}^d x_i\otimes\gamma_i\right\|_{L^1(\Omega;S_1^n)} \leq c_2|||\{x_i\}_{i=1}^d|||^*\,.$$

Then $c_1 = 1/\sqrt{2}$ and $c_2 = 1$.

<u>The Rademacher case</u>:

Theorem 3 (**Haagerup-M. 2007**)
Let $d\,,n\geq 1$ and consider $x_1,\ldots,x_d\in M_n(\mathbb{C})$. Let $\{r_i\}_{1\leq i\leq n}$ be Rademacher functions on $[0,1]$. Then

$$\frac{1}{\sqrt{3}}|||\{x_i\}_{i=1}^d|||^* \;\leq\; \left\|\sum_{i=1}^d x_i\otimes r_i\right\|_{L^1([0,1];S_1^n)} \leq |||\{x_i\}_{i=1}^d|||^* \,.$$

Remark: Let $c_1\,,c_2$ denote the best constants in the inequalities

$$c_1|||\{x_i\}_{i=1}^d|||^* \;\leq\; \left\|\sum_{i=1}^d x_i\otimes r_i\right\|_{L^1([0,1];S_1^n)} \leq c_2|||\{x_i\}_{i=1}^d|||^* \,.$$

Then the following estimates hold

$$\frac{1}{\sqrt{3}}\leq c_1\leq\frac{1}{\sqrt{2}}\,,\quad c_2=1\,.$$

Embedding of OH via Khintchine inequalities for subspaces of $R \oplus C$

It is a classical result that $L^2(\Omega)$ embeds isometrically into $L^1(\Omega)$.

Question: What about the noncommutative analogue of this fact?

The noncommutative analogue of $L^1(\Omega)$ is the predual M_* of a von Neumann algebra M, while the noncommutative analogue of $L^2(\Omega)$ is Pisier's operator Hilbert space OH.

Theorem (**Pisier**): There exists an operator space $OH \subseteq \mathcal{B}(K)$ (where K is a separable Hilbert space) such that

(1) OH is isometric to $l_2(\mathbb{N})$ (as a Banach space)

(2) The canonical identification between OH and $\overline{OH^*}$ (corresponding to the canonical identification between $l_2(\mathbb{N})$ and $\overline{l_2(\mathbb{N})^*}$ is a complete isometry.

Moreover, OH is the unique operator space (up to complete isometry) satisfying (1) and (2).

Some results:

1) OH does not embed (cb-isomorphically) into the predual of any semifinite von Neumann algebra (**Pisier 2004**).

2) OH admits a cb-embedding into the predual of a type III von Neumann algebra (**Junge 2005**). Later, Junge (**2006**) showed that OH cb-embeds into the predual of the hyperfinite type III_1-factor, with cb-isomorphism constant ≈ 200.

(3) OH admits a cb-embedding into the predual of the hyperfinite type III_1-factor, with cb-isomorphism constant $\leq \sqrt{2}$ (**Haagerup-M. 2007**).

The proof is based on Khintchine inequalities for subspaces of $R \oplus C$, proved in the same JFA (2007) paper. The bound of $\sqrt{2}$ comes from the estimate of the constant in this Khintchine-type inequality. (Unfortunately), we showed that this estimate is, again, sharp.

The fascinating question whether the analogue of the classical isometric embedding result of L^2 into L^1 holds (or not) in the noncommutative setting remains open.

Idea of the proof of the OH **embedding**:

Let H be a closed subspace of $R \oplus C$, and associate to it $A \in \mathcal{B}(H)$, $0 \leq A \leq I_H$ so that the operator space structure on H is given by

$$\left\| \sum_{i=1}^{r} x_i \otimes \xi_i \right\|_{M_n(H)} =$$

$$= \max \left\{ \left\| \sum_{i,j=1}^{r} \langle (I_H - A)\xi_i, \xi_j \rangle_H x_i x_j^* \right\|^{\frac{1}{2}} , \left\| \sum_{i,j=1}^{r} \langle A\xi_i, \xi_j \rangle_H x_i^* x_j \right\|^{\frac{1}{2}} \right\}$$

for all $n, r \in \mathbb{N}$, $x_i \in M_n(\mathbb{C})$, $\xi_i \in H$.

Let $\mathcal{A}$ be the CAR-algebra built on H, and let ω_A be the gauge-invariant quasi-free state on $\mathcal{A}$ corresponding to A, that is, for all $n, m \in \mathbb{N}$,

$$\omega_A(a(f_n)^* \ldots a(f_1)^* a(g_1) \ldots a(g_m)) = \delta_{nm} \mathrm{det}(\langle Ag_i, f_j \rangle_H, i, j),$$

for all $f_1, \ldots, f_n, g_1, \ldots, g_m \in H$.

Use Riesz representation theorem to define a map $E_A : \mathcal{A} \to H$ by

$$\langle f, E_A(b) \rangle_H = \omega_A(a(f)b^* + b^* a(f)), \quad f \in H,$$

for all $b \in \mathcal{A}$.

Let π_A be the unital $*$-representation from the GNS construction for $(\mathcal{A}, \omega_A)$. Then E_A extends to a bounded linear operator on

$$M := \overline{\pi_A(\mathcal{A})}^{\mathrm{sot}}.$$

By a result of **Powers-Størmer (1970)**, M is a hyperfinite factor.

Theorem (**Haagerup-M. 2007**)
The map $E_A : \mathcal{A} \to H$ yields a complete isomorphism

$$H \cong \mathcal{A}/\mathrm{Ker}(E_A)$$

with cb-isomorphism constant $\leq \sqrt{2}$. Furthermore, the dual map E_A^* is a complete isomorphism of H^* onto a subspace of M_*.

Now, let P denote the hyperfinite type III_1-factor. Then

$$M \bar{\otimes} P \cong P\,,$$

and hence M_* cb-embeds into P_*. Therefore by the above Theorem, H^* cb-embeds into P_*, with cb-isomorphism constant $\leq \sqrt{2}$.

We deduce that any quotient (and further, any subspace of a quotient) of $(R \oplus C)^*$ cb-embeds into P_*, with cb-isomorphism constant $\leq \sqrt{2}$.

Pisier (2004) showed that OH is a subspace of a quotient of $R \oplus C$. Since OH is self-dual, OH is also a sub-quotient of $(R \oplus C)^*$. Hence OH has this property. □