Coverings of skew products and crossed products by coactions

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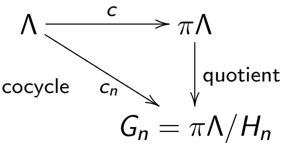
The finite groups

- Let Λ be a k-graph (connected, row-finite, no sources)
- Fix a vertex $v \in \Lambda^0$
- Let $\pi\Lambda = \pi_1(\Lambda, v)$ be the fundamental group of Λ at v [P-Q-Raeburn]
- Choose a cocycle (= functor) $c : \Lambda \to \pi\Lambda$ such that the skew-product k-graph $\Lambda \times_c \pi\Lambda$
  $(\lambda, c(\mu)g)(\mu, g) = (\lambda\mu, g)$
  is the universal covering k-graph of Λ [PQR]
- Suppose we can find a descending chain $\cdots \lhd H_2 \lhd H_1 = \pi\Lambda$ of finite-index normal subgroups
- 

Skew-product coactions

Theorem (PQR)

There is a unique coaction δ^n of G_n on $C^*(\Lambda)$ such that

$$\delta^n(\lambda) = \lambda \otimes c_n(\lambda),$$

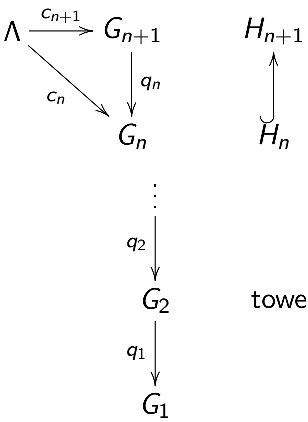
and we have

$$C^*(\Lambda \times_{c_n} G_n) \cong C^*(\Lambda) \times_{\delta^n} G_n$$

Proof.

$\{\lambda \otimes c_n(\lambda)\}$ is a Cuntz-Krieger Λ-family, and $s_{(\lambda,g)} \mapsto (s_\lambda, g)$ gives an isomorphism. □

The inverse limit



tower of surjections of finite groups

$G = \varprojlim (G_n, q_n)$ inverse limit group
compact, totally disconnected

Inverse-limit coaction

Theorem (PQS)

There is a unique coaction δ of $G = \varprojlim(G_n, q_n)$ on $C^*(\Lambda)$ such that

$$\begin{array}{ccc} C^*(\Lambda) & \xrightarrow{\ \delta\ } & M(C^*(\Lambda) \otimes C^*(G)) \\ & {\scriptstyle \delta^n} \searrow & \downarrow {\scriptstyle id \otimes Q_n} \\ & & M(C^*(\Lambda) \otimes C^*(G_n)) \end{array}$$

commutes for all n, and we have

$$C^*(\Lambda) \times_\delta G \cong \varinjlim C^*(\Lambda) \times_{\delta^n} G_n.$$

Here $Q_n : G \to G_n$ is the canonical surjection, and $C^*(\Lambda) \times_{\delta^n} G_n \hookrightarrow C^*(\Lambda) \times_{\delta^{n+1}} G_{n+1}$ in the only way that makes sense.

Proof.

Use Landstad duality and $C(G) = \varinjlim(C(G_n), Q_n^*)$. □

The tower graph

$$\begin{array}{c} \vdots \\ \Big\downarrow{\scriptstyle p_2=\mathrm{id}\times q_2} \\ \Lambda_2 = \Lambda \times_{c_2} G_2 \\ \Big\downarrow{\scriptstyle p_1=\mathrm{id}\times q_1} \\ \Lambda_1 = \Lambda \times_{c_1} G_1 \end{array}$$

tower of coverings of k-graphs

$\varprojlim(\Lambda_n; p_n) = (k+1)$-graph constructed from the sequence of coverings p_n [Kumjian-P-S]

Inverse-limit coaction and tower graph

Theorem (PQS)

$$C^*(\Lambda) \times_\delta G \sim_M C^*(\varprojlim(\Lambda_n; p_n)).$$

Here $\sim_M$ means Morita equivalent.

Proof.

$$\begin{aligned} C^*(\Lambda) \times_\delta G &\cong \varinjlim C^*(\Lambda) \times_{\delta^n} G_n && \text{[PQS]} \\ &\cong \varinjlim C^*(\Lambda \times_{c_n} G_n) && \text{[PQR]} \\ &\sim_M C^*(\varprojlim(\Lambda_n; p_n)) && \text{[KPS]} \end{aligned}$$

□

Example (Bunce-Deddens algebras)

- E = directed graph with one vertex and one loop edge f
- Λ = path category of E, a 1-graph
- $\pi\Lambda = \mathbb{Z} = \langle [f] \rangle$
- $c : \Lambda \to \mathbb{Z}$ cocycle
  $c(f) = 1$
- $H_n = 2^{n-1}\mathbb{Z}$
- $G_n = \mathbb{Z}/2^{n-1}\mathbb{Z}$
- C_j = simple cycle graph with j vertices
- $p_n : \Lambda_{n+1} \to \Lambda_n$, the double covering $C_{2^n} \to C_{2^{n-1}}$

Example

$\varprojlim(\Lambda_n; p_n) \cong$ 2-graph of [P-Raeburn-Rørdam-S] whose C^*-algebra is isomorphic to the Bunce-Deddens algebra of type 2^∞.

Consequently

$$C^*(\Lambda) \times_\delta \varprojlim(G_n, q_n) \sim_M \text{Bunce-Deddens algebra.}$$