

# p-Operator Spaces

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## Operator Spaces

Operator spaces are spaces of operators on Hilbert spaces.

A concrete operator space is a norm closed subspace of  $B(H)$  together with a matrix norm  $\|\cdot\|_n$  on each  $M_n(V)$  given by

$$M_n(V) \subseteq M_n(B(H)) \cong B(\ell_2^n(H)).$$

**Theorem [R 1988]:** Let  $V$  be a Banach space with a norm  $\|\cdot\|_n$  on each matrix space  $M_n(V)$ . Then  $V$  is completely isometrically isomorphic to a concrete operator space if and only if it satisfies

$$\text{M1. } \left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}$$

$$\text{M2. } \|\alpha x \beta\|_n \leq \|\alpha\| \|\|x\|_n\| \|\beta\|$$

for all  $x \in M_n(V)$ ,  $y \in M_m(V)$  and  $\alpha, \beta \in M_n(C) = B(\ell_2^n)$ .

## Completely Bounded Maps

Let  $\varphi : V \rightarrow W$  be a bounded linear map. For each  $n \in \mathbb{N}$ , we can define a linear map

$$\varphi_n : [x_{ij}] \in M_n(V) \rightarrow [\varphi(x_{ij})] \in M_n(W).$$

The map  $\varphi$  is called **completely bounded** if

$$\|\varphi\|_{cb} = \sup\{\|\varphi_n\| : n \in \mathbb{N}\} < \infty.$$

We let  $CB(V, W)$  denote the space of all completely bounded maps from  $V$  into  $W$ , which is again an operator space with matrix norm given by

$$M_n(CB(V, W)) = CB(V, M_n(W)).$$

In particular, the dual space  $V^* = CB(V, \mathbb{C})$  has a natural operator space matrix norm given by

$$M_n(V^*) = CB(V, M_n(\mathbb{C})).$$

In the category of operator spaces, it is important to consider

Matrix Norms and Completely Bounded Maps

## Examples of Operator Spaces:

- $C^*$ -algebras or von Neumann algebras are operator spaces.

In particular, if  $G$  is a locally compact/discrete group,

$$C_\lambda^*(G), C^*(G) \text{ and } VN(G)$$

- Duals  $A^*$  of  $C^*$ -algebras and preduals  $M_*$  of von Neumann algebras

$$A(G) = VN(G)_*, B_\lambda(G) = C_\lambda^*(G)^*, \text{ and } B(G) = C^*(G)^*$$

- Moreover, Herz-Schur multiplier algebra  $M_{cb}A(G) \subseteq CB(A(G), A(G))$ :

$$M_{cb}A(G) = \{\varphi : G \rightarrow C : m_\varphi : \psi \in A(G) \rightarrow \varphi\psi \in A(G) \text{ with } \|m_\varphi\|_{cb} < \infty\}.$$

## Approximation Properties

A C\*-algebra  $A$  is said to be **nuclear** if there exists completely contractive maps  $\varphi_\alpha : A \rightarrow M_{n(\alpha)}$  and  $\psi_\alpha : M_{n(\alpha)} \rightarrow A$  such that

$$\psi_\alpha \circ \varphi_\alpha \rightarrow id_A$$

in the point-norm topology.

A C\*-algebra  $A$  is said to have the **CBAP** (resp. **CCAP**) if there exists finite rank maps  $\varphi_\alpha : A \rightarrow A$  such that  $\|\varphi_\alpha\|_{cb} \leq k$  (resp.  $\|\varphi_\alpha\|_{cb} \leq 1$ ) and  $\varphi_\alpha \rightarrow id_A$  in the point-norm topology.

A C\*-algebra is said to have the **OAP** if for every  $x = [x_{ij}] \in K(\ell_2) \check{\otimes} A$  and  $\varepsilon > 0$ , there exists a finite rank map  $T : A \rightarrow A$  such that

$$\|[T(x_{ij})] - [x_{ij}]\|_{K(\ell_2) \check{\otimes} A} < \varepsilon.$$

A C\*-algebra is said to be **exact** if we have the short exact sequence

$$0 \rightarrow K(\ell_2) \check{\otimes} A \rightarrow B(\ell_2) \check{\otimes} A \rightarrow Q(\ell_2) \check{\otimes} A \rightarrow 0,$$

where  $Q(H) = B(\ell_2)/K(\ell_2)$ .

Let  $G$  be a discrete group.

We have the following implications for group  $C^*$ -algebra  $C_\lambda^*(G)$

Nuclearity  $\Rightarrow$  CCAP/CBAP  $\Rightarrow$  OAP  $\Rightarrow$  Exactness.

If we consider the Fourier algebra  $A(G)$ , then we have

- $C_\lambda^*(G)$  is nuclear  $\Leftrightarrow G$  is amenable  $\Leftrightarrow A(G)$  has a BAI/CAI.
- $C_\lambda^*(G)$  has the CBAP/CCAP  $\Leftrightarrow A(G)$  has the CBAP/CCAP
- $C_\lambda^*(G)$  has the OAP  $\Leftrightarrow A(G)$  has the OAP
- $C_\lambda^*(G)$  is exact  $\Leftrightarrow A(G)$  has ??? .

## Finite Representability in $\{M_n\}$ and in $\{T_n\}$

**Theorem [Pisier 1995]:** A  $C^*$ -algebra  $A$  is exact if and only if it is **finitely representable in  $\{M_n\}$** , i.e. for every f.d. subspace  $E \subseteq A$  and  $\varepsilon > 0$ , there exist  $n(\varepsilon) \in \mathbb{N}$  and  $F \subseteq M_{n(\varepsilon)}$  such that

$$E \stackrel{1+\varepsilon}{\cong}_{cb} F,$$

i.e., there exists a linear isomorphism  $T : E \rightarrow F$  such that

$$\|T\|_{cb} \|T^{-1}\|_{cb} < 1 + \varepsilon.$$

**Question:** It is natural to ask

- whether a  $C^*$ -algebra  $A$  is exact if and only if  $A^*$  is finitely representable in  $\{T_n\}$ ;
- whether  $C_\lambda^*(G)$  is exact if and only if  $A(G)$  is finitely representable in  $\{T_n\}$ .

Actually, it is shown

[Theorem \[Effros- Junge-Ruan 2000\]](#):  $A^*$  (resp.,  $M_*$ ) is finitely representable in  $\{T_n\}$  if and only if  $A$  (resp.,  $M$ ) has QWEP.

[\[Theorem Kirchberg 1993\]](#): QWEP is equivalent to Connes embedding conjecture.

[A. Connes' conjecture 1976](#): Every finite von Neumann algebra with separable predual is  $*$ -isomorphic to a von Neumann subalgebra of the ultrapower of the hyperfinite  $II_1$  factor.

## Operator Algebras on $L_p$ Spaces

Let  $G$  be a locally compact group with a left invariant Haar measure. For  $1 \leq p \leq \infty$ , we may consider

$$L_p(G) = \{f : G \rightarrow \mathbb{C} : \text{measurable } \|f\|_p < \infty\}.$$

For  $p = 1$ , we get the convolution algebra  $L_1(G)$  and

for  $p = \infty$  we get the commutative von Neumann algebra  $L_\infty(G)$ .

For general  $1 < p < \infty$ , we can consider a (regular) representation  $\lambda^p : G \rightarrow B(L_p(G))$  defined by

$$\lambda_s^p(\xi)(t) = \xi(s^{-1}t)$$

for  $\xi \in L_p(G)$ . For  $f \in L_1(G)$ , we have

$$\lambda^p(f) = \int_G f(s) \lambda_s^p ds.$$

We let  $PF_p(G) = \{\lambda^p(f) : f \in L_1(G)\}^{-\|\cdot\|} \subseteq B(L_p(G))$  and

let  $PM_p(G) = \text{span}\{\lambda_s^p : s \in G\}^{-w.o.t} \subseteq B(L_p(G))$ .

We note that  $PM_p(G)$  is a dual space with a predual, the Figà-Talamanca-Herz algebra,

$$A_p(G) = \{f : G \rightarrow C : f(s) = \sum_n \langle \eta_n, \lambda_s^p(\xi_n) \rangle \text{ with } \sum \|\eta_n\|_{L_{p'}} \|\xi_n\|_{L_p} < \infty\}.$$

If we let

$$\Lambda_p : \eta \otimes \xi \in L_{p'}(G) \otimes^\pi L_p(G) \rightarrow \langle \eta, \lambda_s^p(\xi) \rangle .$$

then we can identify

$$A_p(G) = L_{p'}(G) \otimes^\pi L_p(G) / \ker \Lambda_p$$

with the quotient of  $L_{p'}(G) \otimes^\pi L_p(G)$ .

It is known (by Herz 1971) that  $A_p(G)$  is a commutative Banach algebra and  $G$  is amenable if and only if  $A_p(G)$  has a BAI.

## Classical Case

$$C_0(G) \subseteq L_\infty(G)$$

$$L_1(G) \subseteq M(G)$$

## $p=2$ Case

$$C_\lambda^*(G) \subseteq VN(G)$$

$$A(G) \subseteq B_\lambda(G) \subseteq B(G) \\ \subseteq M_{cb}A(G) \subseteq MA(G)$$

## $1 < p < \infty$ Case

$$PF_p(G) \subseteq PM_p(G)$$

$$A_p(G) \subseteq B_{\lambda p}(G) \subseteq B_p(G) \\ \subseteq M_{cb}A_p(G) \subseteq MA_p(G)$$

where  $M(G)$  is the **measure algebra** on  $G$ ,

$MA_p(G)$  is the multiplier algebra of  $A_p(G)$ ,

and  $M_{cb}A_p(G)$  is the **completely bounded multiplier algebra** on  $A_p(G)$ .

For  $p = 2$ ,  $M_{cb}A_2(G) = M_{cb}A(G)$  is the completely bounded **Herz-Schur multiplier algebra** on  $G$ .

What can we say about the corresponding properties on  $p$ -cases ?

## Uniformly Embeddability

Let  $(\Omega, d)$  be a metric space. Then  $(\Omega, d)$  is said to be **uniformly embeddable into a Banach space  $V$**  if there exist a map  $f : \Omega \rightarrow V$  and two monotone increasing functions  $\rho_1 \leq \rho_2$  on  $[0, \infty) \rightarrow [0, \infty)$  such that

$$1) \rho_1(d(x, y)) \leq \|f(x) - f(y)\| \leq \rho_2(d(x, y))$$

$$2) \lim_{t \rightarrow \infty} \rho_1(t) = \infty.$$

It is clear that

Uniformly embeddable into Hilbert spaces

$\Rightarrow$  Uniformly embeddable into  $L_p$  spaces

$\Rightarrow$  Uniformly embeddable into uniformly convex Banach spaces.

**Theorem [Nowak 2005]** Let  $(X, d)$  be a metric space. Then  $(X, d)$  is uniformly embeddable into  $L_p$  space for some  $1 \leq p < 2$  if and only if  $(X, d)$  is uniformly embeddable into  $H$ .

**Remark :** For  $p > 2$ , this is false for general metric space: Johnson and Randrianarivony 2006 showed that  $\ell_p$  does not uniformly embed into  $H$ .

However, it is still an **open question** for finitely generated groups  $(G, d_l)$ .

Let  $G$  be a finitely generated group with a finite generator set  $S = S^{-1}$ . Then we can obtain a length function

$$l(x) = \min\{n : x = g_1 \cdots g_n, g_i \in S\}$$

and a metric  $d_l(x, y) = l(x^{-1}y)$  on  $G$ . Then we can consider uniformly embedding problem for  $(G, d_l)$ .

## p-Operator Spaces

What are p-operator spaces ?

One possible definition is the

“spaces of operators on  $L_p$ -spaces”

i.e.

“subspaces of  $B(L_p(\mu))$ ” .

These are spaces we are interested in this talk !

## General Definition of p-Operator Spaces

Let  $SQ_p$  denote the set of all **subspaces of quotients** of  $L_p$ -spaces, which is equal to the set  $QS_p$  of all **quotients of subspaces** of  $L_p$ -spaces.

Let  $E \in SQ_p$  and  $n \in \mathbb{N}$ , we define  $p$ -column norm on  $E^n = \ell_p^n(E)$ , i.e. we define

$$\|[x_i]\|_p = \left( \sum \|x_i\|^p \right)^{\frac{1}{p}}.$$

We define  $M_n(B(E)) = B(E^n)$ .

A **concrete p-operator space**  $V$  is a norm closed subspace of  $B(E)$  for some  $E \in SQ_p$  together with a matrix norm  $\|\cdot\|_n$  on each  $M_n(V)$  given by

$$M_n(V) \subseteq M_n(B(E)) \cong B(\ell_p^n(E)).$$

**Theorem [Le Merdy 1996]:** Let  $V$  be a Banach space with a norm  $\|\cdot\|_n$  on each matrix space  $M_n(V)$ . Then  $V$  is **completely isometrically isometric** to a  $p$ -concrete operator space if and only if it satisfies

$$\text{M1. } \left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}$$

$$\text{Mp2. } \|\alpha x \beta\|_n \leq \|\alpha\| \|x\|_n \|\beta\|$$

for all  $x \in M_n(V)$ ,  $y \in M_m(V)$  and  $\alpha, \beta \in M_n(C) = B(\ell_p^n)$ .

## p-Completely Bounded Maps

Let  $V$  and  $W$  be  $p$ -operator spaces and let  $\varphi : V \rightarrow W$  be a bounded linear map. Then  $\varphi$  is called  **$p$ -completely bounded** if

$$\|\varphi\|_{pcb} = \sup\{\|\varphi_n\| : n \in \mathbb{N}\} < \infty.$$

We let  $CB_p(V, W)$  denote the space of all  $p$ -completely bounded maps from  $V$  into  $W$ , which is again a  $p$ -operator space with matrix norm given by

$$M_n(CB_p(V, W)) = CB_p(V, M_n(W)).$$

In particular, the dual space  $V^* = CB_p(V, \mathbb{C})$  has a natural  $p$ -operator space matrix norm given by

$$M_n(V^*) = CB_p(V, B(\ell_p^n)).$$

Now we can consider the  $p$ -analogues of operator spaces.

## Second Duals

Let  $V$  be a  $p$ -operator space. We can consider the natural dual matricial norm induced on  $V^{**}$ , i.e. we have

$$M_n(V^{**}) = CB(V^*, B(\ell_p^n)).$$

It is clear that the canonical inclusion

$$\kappa : x \in V \rightarrow \hat{x} \in V^{**}$$

is an isometry and is a complete contraction.

**Theorem [Daws 2007]** For a  $p$ -operator space  $V$ , the canonical inclusion

$$x \in V \hookrightarrow \hat{x} \in V^{**}$$

is a  **$p$ -completely isometric injection** i.e. for every  $[x_{ij}] \in M_n(V)$ , we have

$$\|[x_{ij}]\|_{M_n(V)} = \sup\{\|[\Phi(x_{ij})]\| : \text{all c.c. } \Phi = [\Phi_{kl}] \in M_m(V^*)_1, m \in \mathbb{N}\}$$

if and only if  $V \subseteq B(L_p)$  for some  $L_p$ -space.

Daws showed that

- Every dual p-operator space  $V^*$  is a p-operator subspace of  $B(L_p)$ .
- There exists a finite dimensional p-operator space  $V$  which is not a p-operator subspace of any  $B(L_p)$ , i.e. the p-matrix norm on  $V$  is different from the induced dual matrix norm on  $V^{**} = V$  !
- Another problem is that we do not have the Arveson-Wittstock-Hahn-Banach extension theorem for general p-operator spaces., or for p-operator spaces  $V \subseteq B(L_p)$  .

## Extension Theorem

Hahn-Banach Extension Theorem:

$$\begin{array}{ccc} W & & \\ \uparrow & & \searrow \tilde{\varphi} \\ V & \xrightarrow{\varphi} & \mathbb{C} \end{array}$$

with  $\|\tilde{\varphi}\| = \|\varphi\|$ .

Arveson-Wittstock-Hahn-Banach Extension Theorem:

$$\begin{array}{ccc} W & & \\ \uparrow & & \searrow \tilde{\varphi} \\ V & \xrightarrow{\varphi} & B(H) \end{array}$$

with  $\|\tilde{\varphi}\|_{cb} = \|\varphi\|_{cb}$ .

## Tensor Products for p-operator Spaces

**Le Merdy 1996:** Haagerup tensor product  $\otimes^{h_p}$  is projective, but not necessarily injective.

**Daws 2007:** Projective tensor product  $\otimes^{\wedge p}$  is projective and we have the p-complete isometry

$$CB_p(V \widehat{\otimes}^p W, Z) = CB_p(V, CB_p(W, Z)).$$

In particular,

$$(V \widehat{\otimes}^p W)^* = CB_p(V, W^*).$$

## Injective Tensor Product

We can define the injective tensor product

$$V \check{\otimes}^p W \subseteq CB(V^*, W).$$

If  $W \subseteq B(L_p)$ , then for  $[u_{ij}] \in M_n(V \otimes W)$ , we can write

$$\|[u_{ij}]\|_{V_p} = \sup\{\|[(\Phi \otimes \Psi)(u_{ij})]\| : \Phi \in M_k(V^*)_1, \Psi \in M_l(W^*)\}.$$

**Theorem [An and R 2008]:** Let  $V$  and  $W$  be subspaces of  $B(L_p)$  spaces. Then  $V \check{\otimes}^p W$  is again a subspace of some  $B(L_p)$ .

**Remark:** However, it is still not clear whether  $\check{\otimes}^p$  is injective, due to the lack of p-analogue of the Hahn-Banach extension theorem.

Let  $E \in SQ_p$ . We may define the **column** (respectively, **row**) p-operator space structure  $E^c = B(C, E)$  (respectively,  $E^r = B(E^*, C)$ ). More precisely, we have

$$M_n(E^c) = B(\ell_p^n, \ell_p^n(E)) \text{ (respectively, } M_n(E^r) = B(\ell_p^n(E^*), \ell_p^n)).$$

**Theorem [An and R 2008]** Let  $V \subseteq B(L_p)$ . We have the p-complete isometries

- $V \otimes^{h_p} L_p^c = V \otimes^{\wedge_p} L_p^c$  and  $L_p^r \otimes^{h_p} V = L_p^r \otimes^{\wedge_p} V$
- $L_p^c \otimes^{h_p} V = L_p^c \otimes^{\vee_p} V$  and  $V \otimes^{h_p} L_p^r = V \otimes^{\vee_p} L_p^r$

Combining the above result, we get the following  $p$ -complete isometric isomorphisms

- $T(L_p) \otimes^{\wedge p} V = L_p^r \otimes^{\wedge p} V \otimes^{\wedge p} L_p^c = L_p^r \otimes^{h_p} V \otimes^{h_p} L_p^c$
- $K(L_p) \otimes^{\vee p} V = L_p^c \otimes^{\vee p} V \otimes^{\vee p} L_p^r = L_p^c \otimes^{h_p} V \otimes^{h_p} L_p^r.$

In particular, we get

- $L_p^c \otimes^{h_p} L_p^r = L_p^c \otimes^{\vee p} L_p^r = K(L_p)$  and  $L_p^r \otimes^{h_p} L_p^c = L_p^r \otimes^{\wedge p} L_p^c = T(L_p).$
- $K(L_p)^* = T(L_p)$  and  $T(L_p)^* = B(L_p).$

**Remark:** Though these results are quite similar to operator spaces, the proofs have already need some quite different techniques. For instance, to show  $K(L_p)^* = T(L_p)$  for separable  $L_p$  space, we need the fact:

$L_p$  is separable dual space with RNP and approximation property.

Therefore, every integral map is nuclear, i.e. we have

$$K(L_p)^* = (L_{p'} \otimes^{\epsilon} L_p)^* = I(L_p, L_p) = N(L_p, L_p) = T(L_p).$$

It is known (from definition of  $\widehat{\otimes}^p$ ) that for each  $n \in \mathbb{N}$

$$(T_n \widehat{\otimes}^p V)^* = CB_p(V, B(\ell_p^n)) = M_n(V^*) = M_n \check{\otimes}^p V^*.$$

However, it is quite interesting to show

**Theorem [An and R 2008]:** Let  $V$  be a subspace of  $B(L_p)$ . For each  $n \in \mathbb{N}$ , we have the isometry

$$(M_n \check{\otimes}^p V)^* = T_n \widehat{\otimes}^p V^*.$$

Moreover, we have the complete isometry

$$(K(L_p) \check{\otimes}^p V)^* = T(L_p) \widehat{\otimes}^p V^*.$$

## p-Operator Approximation Property

We can study the corresponding approximation properties for p-operator spaces.

For instance, let  $V$  be a subspace of  $B(L_p)$ , then  $V$  is said to have **p-operator space approximation property (p-OAP)** if for every  $x = [x_{ij}] \in K(\ell_p) \check{\otimes}^p V$  and  $\varepsilon > 0$ , there exists a finite rank map  $T : V \rightarrow V$  such that

$$\|[T(x_{ij})] - [x_{ij}]\|_{K(\ell_p) \check{\otimes}^p V} < \varepsilon.$$

**Theorem [An and R 2008]:** Let  $V$  be a p-operator subspace of some  $B(L_p)$ . Then TFAE:

- 1)  $V$  has the p-OAP;
- 2) the canonical map  $\Phi : V \widehat{\otimes}_p V^* \rightarrow V \check{\otimes}^p V^*$  is injective;
- 3) if  $u \in V \widehat{\otimes}_p V^*$  is such that  $\Phi(u) = 0$ , then  $\text{trace}(u) = 0$ .

Now let  $G$  a locally compact group. Then  $M_{cb}A_p(G) = \{\varphi : G \rightarrow \mathbb{C}, m_\varphi : \psi \in A_p(G) \rightarrow \varphi\psi \in A_p(G) \text{ with } \|m_\varphi\|_{cb} < \infty\}$ .

**Theorem [Daws 2007]:** A continuous function  $\varphi : G \rightarrow \mathbb{C}$  is a cb multiplier of  $A_p(G)$  if and only if there exist bounded maps  $\xi : G \rightarrow E$  and  $\eta : G \rightarrow E'$  (with  $E$  in  $SQ_p$  and  $E'$  in  $SQ_{p'}$ ) such that

$$\varphi(st^{-1}) = \langle \eta(s), \xi(t) \rangle .$$

Moreover we have  $\|m_\varphi\|_{pcb} = \inf\{\|\eta\|_\infty \|\xi\|_\infty\}$ .

**Proposition [Miao and R 08]:**  $M_{cb}A_p(G)$  is a dual space with a predual  $Q_p(G)$ .

Therefore, we can define **p-AP** for  $G$ :

1 is contained in the  $\sigma(M_{cb}A_p(G), Q_p(G))$  closure of  $A_p(G)$  in  $M_{cb}A_p(G)$ .

Now we are working on the p-OAP for  $FP_p(G)$  and  $A_p(G)$ . Most of results by Haagerup-Kraus 1994 can be generalized to p-case.

Thank you for your attention !