Bundles of C^*-algebras and a theorem of Eilenberg and Steenrod

Claude Schochet

joint with J. Klein, N.C. Phillips, Sam Smith

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Notation

X compact space, separable

G topological group (e.g. the group of unitaries of a separable unital C^*-algebra.)

$\zeta : G \longrightarrow T \longrightarrow X$ principal G-bundle (or more generally, principal G-fibration)

A separable unital C^*-algebra

$G \longrightarrow Aut(A)$ faithful representation

$A \longrightarrow A \times_G T \longrightarrow X$ associated bundle of A-algebras

$A_\zeta = \Gamma(X, A \times_G T)$ the C^*-algebra of sections

$UA_\zeta = \Gamma(X, UA \times_G T)$ the unitary group of the C^*-algebra of sections

Theorem A. *(Eilenberg-Steenrod)* [1, Chapter X, Theorem 10.1] *Every compact space X is homeomorphic to the inverse limit of an inverse system of finite simplicial complexes. If the space X is separable then the system may be taken to be a sequence $X \cong \varprojlim X_j$.*

Thus any separable commutative C^*-algebra $C(X)$ is the direct limit of a sequence of commutative C^*-algebras $C(X_j)$ where X_j is a finite complex.

What about bundles? We have

$$\begin{array}{ccc} A \times_G T & \longrightarrow & ??? \\ \downarrow & & \downarrow \\ X & \longrightarrow & X_j \end{array}$$

No obvious construction.

Theorem B. *(Eilenberg and Steenrod* [1] *Ch. X, Theorem 11.9) Suppose that X is the inverse limit of compact spaces $\{X_j\}$ and Y is a simplicial complex. Let $f : X \longrightarrow Y$ be a map. Then there exists an index m and a map*

$$f_m : X_m \longrightarrow Y$$

such that the composite

$$X \xrightarrow{\pi_m} X_m \xrightarrow{f_m} Y$$

is homotopic to f.

Let F denote function space of all maps and $F_\bullet$ denote based maps.

Theorem C. *(built on Spanier (*[4] *Thm. 13.4) Suppose that X is a compact space written as an inverse limit of compact spaces X_j and Y is a CW complex. Then:*

(1) *There is a natural isomorphism*
$$\varinjlim \pi_*(F_\bullet(X_j, Y)) \cong \pi_*(F_\bullet(X, Y)).$$

(2) *If Y is simple then there is a natural isomorphism*
$$\varinjlim \pi_*(F(X_j, Y)) \cong \pi_*(F(X, Y)).$$

Recall that $A_\zeta = \Gamma(X, A \times_G T)$. Need to assume that principal G-bundle ζ is standard; that is, there is a pullback diagram

$$\begin{array}{ccc} T & \longrightarrow & EG \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BG \end{array}$$

with G (and hence BG) the homotopy type of a CW complex. Pullbacks commute with associated bundles and hence there is a pullback diagram

$$\begin{array}{ccc} A \times_G T & \longrightarrow & A \times_G EG \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BG \end{array}$$

and similarly

$$\begin{array}{ccc} UA \times_G T & \longrightarrow & UA \times_G EG \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BG. \end{array}$$

Now factor f by Theorem B to obtain a pullback diagram

$$\begin{array}{ccccc} T & \longrightarrow & T_m & \longrightarrow & EG \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & X_m & \longrightarrow & BG \end{array}$$

with G (and hence BG) the homotopy type of a CW complex. Pullbacks homotopy commute with associated bundles and hence there is a pullback diagram

$$\begin{array}{ccccc} A \times_G T & \longrightarrow & A \times_G T_m & \longrightarrow & A \times_G EG \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & X_m & \longrightarrow & BG \end{array}$$

and similarly

$$\begin{array}{ccccc} UA \times_G T & \longrightarrow & UA \times_G T_m & \longrightarrow & UA \times_G EG \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & X_m & \longrightarrow & BG. \end{array}$$

Restrict attention to the cofinal family $\{X_k\}$ over X_m. Define

$$\zeta_k \qquad G \longrightarrow T_k \longrightarrow X_k$$

to be the pullback of the universal bundle by the map $X_k \to X_m \to BG$ and

$$A_{\zeta_k} = \Gamma(X_k, A \times_G T_k).$$

Then there are natural bundle maps $\zeta_j \to \zeta_k$ for $j > k$ and hence natural maps $A_{\zeta_k} \to A_{\zeta_j}$ that induce

$$\varinjlim A_{\zeta_k} \longrightarrow A.$$

Theorem D. *With the notation above, there is a natural isomorphism*

$$\varinjlim \pi_*(UA_{\zeta_j}) \xrightarrow{\cong} \pi_*(UA).$$

Note that this implies immediately the (much weaker) fact that there is a natural isomorphism

$$\varinjlim K_*(A_{\zeta_j}) \xrightarrow{\cong} K_*(A).$$

References

[1] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton University Press, Princeton, 1952. MR 0050886 (14,398b)

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[4] E. Spanier, *Borsuk's cohomotopy groups*, Ann. Math. **50** (1949), 203 - 245. MR 58 #24264

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[6] N. Steenrod, *The topology of fibre bundles*, Princeton University Press, Princeton, 1951 MR 0039258