Is every CP-semigroup a “part of” an E-semigroup?

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Background: CP and E-semigroups

Let H be a **separable** Hilbert space.

- **CP-semigroup** - A semigroup $\Theta = \{\Theta_t\}_{t \geq 0}$ ($\Theta_s \circ \Theta_t = \Theta_{s+t}$ and $\Theta_0 = I$) of normal, contractive, completely positive maps on a v.N. algebra $\mathcal{M} \subseteq B(H)$, continuous in the following sense: for all $h, g \in H$, and all $a \in \mathcal{M}$,

$$\lim_{t \to t_0} \langle \Theta_t(a)h, g \rangle = \langle \Theta_{t_0}(a)h, g \rangle .$$

- **E-semigroup** a CP-semigroup with every element a $*$-endomorphism.
- **CP$_0$-semigroup** - a CP-semigroup with $\Theta_t(I) = I$.
- **E$_0$-semigroup** - an E-semigroup with $\Theta_t(I) = I$. (Arveson, Powers,...)

Background: E-dilation of CP-semigroups

Given a semigroup $\Theta = \{\Theta_t\}_{t \geq 0}$ of CP maps acting on $\mathcal{M} \subseteq B(H)$, an **E-dilation** of Θ is a triple $(\alpha, \mathcal{R}, K)$, where

- $K \supseteq H$ is a Hilbert space,
- $\mathcal{R} \subseteq B(K)$ is a v.N. algebra such that $\mathcal{M} = P_H \mathcal{R} P_H$,
- $\alpha = \{\alpha_t\}_{t \geq 0}$ is an E-semigroup such that

  $$\Theta_t(P_H T P_H) = P_H \alpha_t(T) P_H,$$

  for all $T \in \mathcal{R}, t \geq 0$.
- If Θ is unital (i.e., a CP_0-semigroup) we would like α to be unital as well.

E-dilation of CP-semigroups

Theorem (Bhat, SeLegue, Bhat-Skeide, Muhly-Solel, Arveson)

Let $\Theta = \{\Theta_t\}_{t \geq 0}$ *be a CP-semigroup acting on* $\mathcal{M} \subseteq B(H)$*. Then* Θ *has an E-dilation.*

That is, There is a Hilbert space $K \supseteq H$*, a v.N. algebra* $\mathcal{R} \subseteq B(K)$ *such that* $\mathcal{M} = P_H \mathcal{R} P_H$*, and an E-semigroup* $\{\alpha_t\}_{t \geq 0}$ *on* $\mathcal{R}$ *such that for all* $T \in \mathcal{R}, t \geq 0$

$$\Theta_t(P_H T P_H) = P_H \alpha_t(T) P_H.$$

Remarks

- Unitality.
- Minimality (α cannot be restricted nor compressed to a smaller dilation of Θ).
- Uniqueness (up to **conjugacy**).
- if $\mathcal{M} = B(H)$ then $\mathcal{R} = B(K)$.

Problem

Can similar results be obtained if $\mathbb{R}_+$ is replaced by some semigroup $\mathcal{S} \subseteq \mathbb{R}_+^k$?

- The case $\mathcal{S} = \mathbb{N}$ corresponds to a single CP map (OK).
- The case $\mathcal{S} = \mathbb{N}^k$ corresponds to k commuting CP maps. $k = 2$ was solved by Bhat (for $\mathcal{M} = B(H)$) and later by Solel (general $\mathcal{M}$). $k > 2$ is not expected to be true in general. However...
- The case $\mathcal{S} = \mathbb{R}_+^k$ corresponds to k commuting CP-semigroups. **We treated mainly the case** $k = 2$. However...

The main results - 1

Theorem (Unital case, general $\mathcal{M}$)

Let $\Phi = \{\Phi_t\}_{t\geq 0}$, $\Psi = \{\Psi_t\}_{t\geq 0}$ be two **strongly commuting** *CP_0-semigroups on $\mathcal{M} \subseteq B(H)$. Then the semigroup Θ (over $\mathbb{R}^2_+$) of CP-maps given by*

$$\Theta_{(s,t)} = \Phi_s \circ \Psi_t$$

has a minimal E_0-dilation.
That is, there is a Hilbert space $K \supseteq H$, a v.N. algebra $\mathcal{R} \subseteq B(K)$ such that $\mathcal{M} = P_H \mathcal{R} P_H$, and there are two commuting E_0-semigroups α and β on $\mathcal{R}$ such that

$$\Phi_s \circ \Psi_t(P_H T P_H) = P_H \alpha_s \circ \beta_t(T) P_H,$$

for all $T \in \mathcal{R}$, $s, t \in \mathbb{R}_+$.

The main results - 2

Regarding the proof

- Unitality is essential.
- Strong commutativity (whatever it is...) is essential. However, if $\mathcal{M} = M_n(\mathbb{C})$, then all commuting pairs of CP-semigroups satisfy strong commutativity.
- $k = 2$ is used.

What can we say about the structure of the dilation?

- The dilation is minimal (but not expected to be unique).
- If $\mathcal{M} = B(H)$ then $\mathcal{R} = B(K)$.
- α and β are usually **cocycle conjugate** to the minimal dilations of Φ and Ψ.
- This raises more questions (classification, existence of new E_0-semigroups).

The main results - 3

Theorem (Non-unital case, $\mathcal{M} = B(H)$)

Let $\Phi = \{\Phi_t\}_{t\geq 0}$, $\Psi = \{\Psi_t\}_{t\geq 0}$ *be two* **strongly commuting** *CP-semigroups on* $B(H)$. *Then the semigroup* Θ *(over* $\mathbb{R}_+^2$*) of CP-maps given by*

$$\Theta_{(s,t)} = \Phi_s \circ \Psi_t$$

has a minimal E-dilation.

That is, there is a Hilbert space $K \supseteq H$ *and there are two commuting E-semigroups* α *and* β *on* $B(K)$ *such that*

$$\Phi_s \circ \Psi_t(P_H T P_H) = P_H \alpha_s \circ \beta_t(T) P_H,$$

for all $T \in B(K)$, $s, t \in \mathbb{R}_+$.

The main results - 4

Regarding the proof

- Dropping the assumption of unitality leads to a significant change in the proof.
- Strong commutativity (whatever it is...) is still essential. For $\mathcal{M} = M_n(\mathbb{C})$ we have a complete answer.
- In fact, we have a complete answer for any finite dimensional C*-algebra.
- The assumption $\mathcal{M} = B(H)$ is essential.
- $k > 2$ is not expected to be true.

What can we say about the structure of the dilation?

That's a good question.

So what is strong commutativity?

Two CP maps Φ and Ψ on $\mathcal{M}$ are said to **commute strongly** if there is a unitary

$$u : \mathcal{M} \otimes_\Phi \mathcal{M} \otimes_\Psi H \to \mathcal{M} \otimes_\Psi \mathcal{M} \otimes_\Phi H$$

satisfying:

1. $u(a \otimes_\Phi I \otimes_\Psi h) = a \otimes_\Psi I \otimes_\Phi h$, for all $a \in \mathcal{M}, h \in H$.
2. $u(ca \otimes_\Phi b \otimes_\Psi h) = (c \otimes I \otimes I)u(a \otimes_\Psi b \otimes_\Phi h)$ for all $a, b, c \in \mathcal{M}, h \in H$.
3. $u(a \otimes_\Phi b \otimes_\Psi dh) = (I \otimes I \otimes d)u(a \otimes_\Psi b \otimes_\Phi h)$ for all $a, b \in \mathcal{M}, h \in H$ and $d \in \mathcal{M}'$.

Two semigroups $\Phi = \{\Phi_t\}_{t \geq 0}$, $\Psi = \{\Psi_t\}_{t \geq 0}$ are said to **commute strongly** if for all s, t the maps Φ_s and Ψ_t commute strongly.

Equivalent definition of strong commutativity

The *GNS*-representation of a CP map Φ on $\mathcal{M}$ is a pair (E, ξ) satisfying

$$\Phi(a) = \langle \xi, a\xi \rangle \ , \ a \in \mathcal{M}. \tag{1}$$

$E = \mathcal{M} \otimes_\Phi \mathcal{M}$, the Hilbert W*-correspondence with inner product

$$\langle a \otimes_\Phi b, c \otimes_\Phi d \rangle = b^* \Phi(a^* c) d,$$

and $\xi = 1 \otimes_\Phi 1 \in E$.

Theorem (Alternative definition)

Let Φ and Ψ be CP maps with GNS-representations (E, ξ) and (F, η), respectively. Φ and Ψ commute strongly if and only if there exists an isomorphism of W-correspondences*

$$U : E \otimes F \to F \otimes E$$

$$U : \xi \otimes \eta \mapsto \eta \otimes \xi.$$

Strong commutativity in $B(H)$

Theorem (Solel)

Let $(T_1, \ldots, T_n)$ and $(S_1, \ldots, S_m)$ ($m, n \in \mathbb{N} \cup \{\infty\}$) be two ℓ^2-independent row contractions. The CP maps

$$\Phi(a) = \sum_i T_i a T_i^* \quad , \quad \Psi(a) = \sum_j S_j a S_j^* \, ,$$

commute strongly if and only if there is an $mn \times mn$ unitary matrix

$$U = \left(U_{(i,j)}^{(k,l)} \right)_{(i,j),(k,l)}$$

such that for all i, j,

$$T_i S_j = \sum_{(k,l)} U_{(i,j)}^{(k,l)} S_l T_k .$$

Strong commutativity in ℓ^∞

$\mathcal{M} = \mathbb{C}^n, \ell^\infty$ acting as diagonal matrices on $H = \mathbb{C}^n, \ell^2$.
A unital, CP map is a stochastic matrix P:

$$p_{ij} \geq 0 \ , \ i, j$$

and

$$\sum_j p_{ij} = 1 \ , \ i.$$

Two matrices P and Q strongly commute $\Leftrightarrow$ for all i, k,

$$|\{j : q_{kj} p_{ji} \neq 0\}| = |\{j : p_{kj} q_{ji} \neq 0\}|.$$

All commuting positive matrices do so strongly.
Example for non-strong commutation:

$$P = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \ , \ Q = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}.$$

Proof outline – the nonunital case

Given: a CP-semigroup $\Theta = \{\Theta_s\}_{s\in\mathbb{R}^2_+}$ on $B(H)$.

Step 1:

Construct p.s. $X = \{X(s)\}_{s\in\mathbb{R}^2_+}$ and representation T

$$\Theta_s(a) = \widetilde{T}_s \left(I_{X(s)} \otimes a\right) \widetilde{T}_s^*.$$

Step 2:

Dilate $T\big|_{\mathbb{D}^2_+}$ to an isometric V of $\{X(s)\}_{s\in\mathbb{D}^2_+}$ on $K \supseteq H$.

Step 3:

$$\gamma_s(a) := \widetilde{V}_s \left(I_{X(s)} \otimes a\right) \widetilde{V}_s^* \ , \ s \in \mathbb{D}^2_+, a \in B(K).$$

Continue the semigroup $\{\gamma_s\}_{s\in\mathbb{D}^2_+}$ to $\mathbb{R}^2_+$. Check that it is a minimal dilation of Θ.

Step 1: Product system representations

c.c. representation

of a Hilbert space E on a Hilbert space H is a linear, completely contractive map $T : E \to B(H)$.
$\widetilde{T} : E \otimes H \to H$ is defined: $\widetilde{T}(\xi \otimes h) = T(\xi)h$.

Product system of Hilbert spaces

$X = \{X(s)\}_{s \in \mathcal{S}}$, $X(0) = \mathbb{C}$,

$$X(s) \otimes X(t) \cong X(s+t).$$

Product system representation

A family $T = \{T_s\}_{s \in \mathcal{S}}$, T_s is a c.c. representation of $X(s)$ on H,

$$T_{s+t}(x_s \otimes x_t) = T_s(x_s) T_t(x_t).$$

Step 1 (cont.): Representing the CP-semigroup

- M-S constructed $E = \{E(t)\}_{t\geq 0}$, $F = \{F(t)\}_{t\geq 0}$ with representation $T^E : E \to B(H)$, $T^F : F \to B(H)$ such that

$$\Phi_t(a) = \widetilde{T}_t^E \left(I_{E(t)} \otimes a\right) (\widetilde{T}_t^E)^*,$$

$$\Psi_t(a) = \widetilde{T}_t^F \left(I_{F(t)} \otimes a\right) (\widetilde{T}_t^F)^*.$$

- Define $X = \{X(s,t)\}_{(s,t)\in\mathbb{R}_+^2}$ by

$$X(s,t) = E(s) \otimes F(t).$$

- Define $T = \{T_{(s,t)}\}_{(s,t)\in\mathbb{R}_+^2}$ by

$$T_{(s,t)}(e_s \otimes f_t) = T_s^E(e_s) T_t^F(f_t).$$

Very technical. Need **strong commutativity**.

Step 2: Dilating the product system representation

The product system X and representation T constructed in step 1 satisfy

$$\Phi_s(\Psi_t(a)) = \widetilde{T}_{(s,t)}(I_{X(s,t)} \otimes a)\widetilde{T}^*_{(s,t)}.$$

$\widetilde{T}$ is isometric $\Rightarrow$ the CP-semigroup is actually an E-semigroup.
$\widetilde{T}$ is coisometric $\Leftrightarrow$ the CP-semigroup is a CP_0-semigroup.

Isometric representation:

a representation T such that $\widetilde{T}_s$ is an isometry for all $s \in \mathbb{R}^2_+$.

Fully-coisometric representation:

a representation T such that $\widetilde{T}_s$ is a coisometry for all $s \in \mathbb{R}^2_+$.

Step 2: Dilating the representation (cont.)

Theorem

Let $X = \{X(s)\}_{s\in\mathbb{D}^2_+}$ be a product system, and let T be a c.c. representation of X on H. Then there exists a Hilbert space $K \supseteq H$ an isometric representation V of X on K, such that

1. $P_H V_s(x)\big|_H = T_s(x)$ *for all* $s \in \mathbb{R}^k_+$, $x \in X(s)$.
2. $P_H V_s(x)\big|_{K\ominus H} = 0$ *for all* $s \in \mathcal{S}$, $x \in X(s)$.
3. $K = \bigvee\{V(x)h : x \in X, h \in H\}$.

Proof.

Idea: positive definite functions and a trick of Ptak, who proved the analogous result for two-parameter semigroup of contractions. □

Step 3: Putting the pieces together

Given: two strongly commuting CP_0-semigroups $\Phi = \{\Phi_t\}_{t\geq 0}$, $\Psi = \{\Psi_t\}_{t\geq 0}$ on $B(H)$.

- Step 1:
$$\Phi_s(\Psi_t(a)) = \widetilde{T}_{(s,t)}(I_{X(s)} \otimes a)\widetilde{T}^*_{(s,t)}.$$
- Step 2: construct isometric dilation V of $T\big|_{\mathbb{D}^2_+}$ on $K \supseteq H$.
- Step 3:
$$\gamma_{(s,t)}(b) = \widetilde{V}_{(s,t)}(I_{X(s)} \otimes b)\widetilde{V}^*_{(s,t)}\ ,\ b \in B(K),$$
for all $s, t \in \mathbb{D}^2_+$.

Putting $\alpha_t = \gamma_{(t,0)}$ and $\beta_t = \gamma_{(0,t)}$ we obtain a commuting pair of semigroups of $*$-automorphisms which dilate $\Phi\big|_{\mathbb{D}^2_+}$, $\Psi\big|_{\mathbb{D}^2_+}$.

Step 3: Putting the pieces together (cont.)

Problem

Can α and β be continued continuously to E-semigroups (parameterized by $\mathbb{R}_+$?). If so, is the dilation minimal?

Yes:

Put $\mathcal{A}_0 = \bigcup_{s \in \mathbb{D}_+^2} \gamma_s(B(H))$, $\mathcal{A} = C^*(\mathcal{A}_0)$.

1. Using minimality of V one shows that $K = [\mathcal{A}H]$, $B(K) = W^*(\mathcal{A})$ (minimality).
2. Then one shows that for all $a \in \mathcal{A}$, $h, g \in K$

   $$\lim_{\mathbb{D}_+ \ni t \to 0} \langle \alpha_t(a)h, g \rangle = \langle ah, g \rangle,$$

   and also

   $$\lim_{\mathbb{D}_+ \ni t \to 0} \langle \alpha_t(1)h, g \rangle = \langle h, g \rangle.$$

3. $\mathcal{A}$ is irreducible and contains $B(H)$, hence contains $\mathcal{K}(K)$.

Digression: Extending densely parameterized semigroups

Theorem (Essentially due to Arveson, SeLegue)

Let $\mathcal{S}$ be a dense subsemigroup of $\mathbb{R}_+$, and let $\Theta = \{\Theta_s\}_{s\in\mathcal{S}}$ be a semigroup of contractive, normal, positive maps on $B(H)$. Assume that for all compact $x \in B(H)$ and every normal state ρ

$$\lim_{\mathcal{S}\ni s\to 0} \rho(\Theta_s(x)) = \rho(x) \;\text{ and }\; \lim_{\mathcal{S}\ni s\to 0} \rho(\Theta_s(1)) = \rho(1).$$

Then Θ can be extended to a semigroup $\hat{\Theta} = \{\hat{\Theta}_t\}_{t\geq 0}$ of contractive, positive, normal maps on $B(H)$ such that $\hat{\Theta}_s = \Theta_s$ for all $s \in \mathcal{S}$ and such that

$$\lim_{t\to t_0} \rho(\hat{\Theta}_t(a)) = \rho(\hat{\Theta}_{t_0}(a)) \;\text{ for all }\; a \in B(H)\,,\; t_0 \geq 0.$$

If Θ consists of CP-maps/$*$-endomorphisms/unital maps then so does $\hat{\Theta}$.

Step 3: Putting the pieces together (finished)

The previous theorem shows that α and β can be extended continuously to semigroups over $\mathbb{R}_+$. These two semigroups commute, and $(\alpha, \beta, B(K), K)$ is an E-dilation of Φ, Ψ:

$$\Phi_s \circ \Psi_t(P_H T P_H) = P_H \alpha_s \circ \beta_t(T) P_H,$$

for all $T \in B(K)$, $s, t \in \mathbb{R}_+$.

A few words about the unital case

- Step 1 is carried out in the same manner.
- Step 2: Since the representation T is fully-coisometric, this is easier. The dilation is obtained over all the points $(s,t) \in \mathbb{R}_+^2$.
- Step 3 is easier. One of the reasons: no need to continue the semigroups.
- In fact, we proved the existence of an isometric dilation of a fully-coisometric representation of any product system over a subsemigroup of $\mathbb{R}_+^k$, for any k.
- The "only" thing that remains to do to obtain an E_0-dilation of k (strongly) commuting CP_0-semigroups is to construct a product system and representation that represents the CP_0-semigroups.