

Graph-, Exel-Laca-, and ultragraph algebras coincide up to Morita equivalence.

Joint with Katsura, Muhly, and Tomforde.

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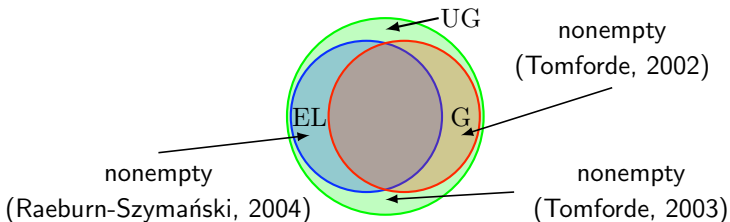


Part 1: Background



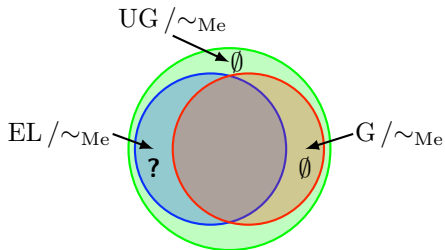
Context

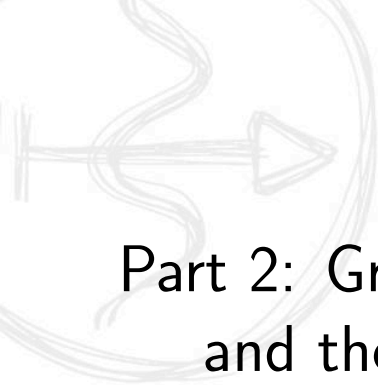
- Graph algebras and Exel-Laca algebras were introduced (late 90's) as infinitely-generated generalisations of Cuntz-Krieger algebras.
- There are graph algebras which are not isomorphic to any Exel-Laca algebra and vice-versa.
- Ultragraph algebras were introduced by Tomforde (2003) to unify the two. There are ultragraph algebras which belong to neither subclass.



What about Morita equivalence?

- The extent to which G , EL , and UG are separated by *isomorphism* classes of C^* -algebras is resolved.
- What about *Morita equivalence* classes?
- $G / \sim_{Me} \subseteq EL / \sim_{Me}$ (Fowler-Laca-Raeburn 2000).
- $EL / \sim_{Me} = UG / \sim_{Me}$ (Tomforde, 2002).
- But $EL / \sim_{Me} \subseteq G / \sim_{Me}$? Yes (Katsura-Muhly-S-Tomforde).





Part 2: Graphs, ultragraphs, and their C^* -algebras



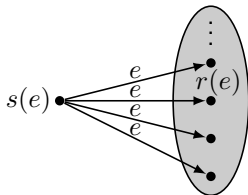
Graphs and graph C^* -algebras

- A directed graph E consists of a countable set E^0 of vertices, a countable set E^1 of edges, and maps $r, s : E^1 \rightarrow E^0$.
- The edge $e \in E^1$ points from $s(e)$ to $r(e)$: $s(e) \bullet \xrightarrow{e} \bullet r(e)$
- Let $E = (E^0, E^1, r, s)$ be a directed graph. Then $C^*(E)$ is generated by
 - 1 mutually orthogonal projections $\{p_v : v \in E^0\}$ and
 - 2 partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal final projections such that $s_e^* s_e = s_{r(e)}$, and
 - 3 $p_v = \sum_{s(e)=v} s_e s_e^*$ whenever $0 < |s^{-1}(v)| < \infty$.



Ultragraphs

- An ultragraph consists of a countable set G^0 of vertices, a countable set \mathcal{G}^1 of edges and maps $s : \mathcal{G}^1 \rightarrow G^0$, $r : \mathcal{G}^1 \rightarrow \mathcal{P}(G^0) \setminus \emptyset$.
- The edge e points from the vertex $s(e)$ to all the vertices in $r(e)$:



- Write \mathcal{G}^0 for smallest subset of $\mathcal{P}(G^0)$ containing singletons, each $r(e)$, and closed under finite intersections, finite unions, and relative complements.

Ultragraph C^* -algebras

- Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be an ultragraph.
- The ultragraph C^* -algebra $C^*(\mathcal{G})$ is generated by
 - 1 projections $\{p_A : A \in \mathcal{G}^0\}$ satisfying
 - $p_A p_B = p_{A \cap B}$,
 - $p_{A \cup B} = p_A + p_B - p_{A \cap B}$, and
 - $p_{A \setminus B} = p_A - p_{A \cap B}$; and
 - 2 partial isometries $\{s_e : e \in \mathcal{G}^1\}$ with mutually orthogonal final projections such that $s_e^* s_e = p_{r(e)}$, and
 - 3 $p_v = \sum_{s(e)=v} s_e s_e^*$ whenever $0 < |s^{-1}(v)| < \infty$.
- A graph is an ultragraph in which each $r(e)$ is a singleton.



Graph C^* -algebras Vs Ultragraph C^* -algebras

- These look quite different.
- In particular, the spectrum of the commutative C^* -subalgebra $\overline{\text{span}}\{p_A : A \in \mathcal{G}^0\}$ of an ultragraph C^* -algebra could be quite complicated, whereas the spectrum of $\overline{\text{span}}\{p_v : v \in E^0\}$ in a graph C^* -algebra is just a copy of E^0 .
- But there's some hope: every AF algebra is Morita equivalent to a graph C^* -algebra.
- In particular, $\overline{\text{span}}\{p_A : A \in \mathcal{G}^0\}$ is Morita equivalent to a graph C^* -algebra because it is the inductive limit of the C^* -algebras generated by finitely many $p_{r(e)}$ and finitely many $p\{v\}$.
- This is the germ of the answer to our original question.





Part 3: Graph from ultragraph: the construction



Encoding \mathcal{G}^0 .

- Fix an ultragraph \mathcal{G} .
- Want a graph E so that $C^*(\mathcal{G})$ is a full corner of $C^*(E)$.
- First step is to worry about how to encode $B := C^*(\{p_A : A \in \mathcal{G}^0, |A| = \infty\})$.
- For $e \in \mathcal{G}^1$, let $r(e)^1 = r(e)$, and $r(e)^0 = G^0 \setminus r(e)$.
 - list $\mathcal{G}^1 = \{e_1, e_2, \dots\}$.
 - let $\Delta^0 = \bigcup_{n=1}^{\infty} \left\{ \omega \in \{0, 1\}^n \setminus \{0^n\} : \left| \bigcap_{i=1}^n r(e_i)^{\omega_i} \right| = \infty \right\}$
 - $\Delta^1 = \{e_\omega : \omega \in \Delta, \omega \neq (0^{n-1}, 1)\}$.
 - $r(e_\omega) = \omega$; $s(e_\omega) = (\omega_1, \dots, \omega_{n-1})$.
 - Δ is like a Bratteli diagram for the infinite elements of \mathcal{G}^0 .
- $E^0 = G^0 \sqcup \Delta^0$, but E^1 requires some more work.



Connecting G^0 to Δ

- Now to replace ultra-edges with edges.
- If $r(e_n)$ is finite, draw an edge e_n^v in E from $s(e_n)$ to v for each $v \in r(e_n)$.
- When $|r(e_n)| = \infty$, we must work harder:
 - Draw an edge from $s(e_n)$ to each ω in Δ_n with $\omega_n = 1$.
 - Choose an assignment of an element $\sigma(v) \in \Delta$ to each $v \in G^0$ belonging to an infinite set of the form $\bigcap_{i=1}^{n+k} r(e_i)^{\omega_i}$ such that
 - $v \in \bigcap_{i=1}^{n+k} r(e_i)^{\sigma(v)_i}$ for all v ; and
 - each $\omega \in \Delta$ is equal to $\sigma(v)$ for at most finitely many v .
 - For each n , draw edges e_n^v as above for those $v \in r(e_n)$ with $|\sigma(v)| < n$.
 - For the remaining $v \in r(e_n)$, draw an edge from ω to v .
- Now E^1 is Δ^1 plus the stuff constructed above.



Results.

Theorem 3.1

There is a subset V of E^0 so that $P = \sum_{v \in V} p_v$ is full in $MC^(E)$ and $PC^*(E)P \cong C^*(\mathcal{G})$.*

Corollary 3.2

We have $UG / \sim_{Me} \subseteq G / \sim_{Me}$.

- Can relate path-structure of E to that of \mathcal{G} .
- \mathcal{G} satisfies Condition (K) iff E does, so $\text{RR}(C^*(\mathcal{G})) = 0$ iff \mathcal{G} satisfied Condition (K).
- Can describe quotients of $C^*(\mathcal{G})$ by gauge-invariant ideals as full corners of graph algebras.

