Spectral Shift and Trace Inequalities

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Small perturbations

$$\begin{array}{ccc} & V = V^* & \\ H_0 = H_0^* & \xrightarrow{\hspace{3em}} & H_0 + V \\ & ? & \\ \sigma(H_0) & \xrightarrow{\hspace{3em}} & \sigma(H_0 + V) \end{array}$$

Originally, $\sigma(V)$ is "much thinner" than $\sigma(H_0)$

"Thick" portion of $\sigma(H_0)$ does not change:

V is compact $\Rightarrow \sigma_{ess}(H_0) = \sigma_{ess}(H_0 + V)$

V is in the trace class $\Rightarrow \sigma_{ac}(H_0) = \sigma_{ac}(H_0 + V)$

"Thin" portion of $\sigma(H_0)$ may change. The Birman-Schwinger principle:

$H_0 > 0$, $V = -K^2$, $K = K^* \Rightarrow \sigma(H_0 + V) \cap \mathbb{R}_- = \sigma_{disc}(H_0 + V)$

$$\mathrm{tr}[E_{H_0+V}(\mathbb{R}_-)] = \mathrm{tr}[E_{KH_0^{-1}K}((1,\infty))]$$

Spectral shift functions (SSF) I

How does the spectrum shift (relative to $\lambda \in \mathbb{R}$)?

V is in the trace class, $E_H(\lambda) := E_H((-\infty, \lambda))$

what portion (Krein's SSF '52):

$$\text{“}\mathrm{tr}\big[E_{H_0}(\lambda) - E_{H_0+V}(\lambda)\big]\text{”} =: \xi_{H_0+V,H_0}(\lambda)$$

$$\mathrm{tr}\big[f(H_0+V) - f(H_0)\big] = \int_{-\infty}^{\infty} f'(\lambda)\xi_{H_0+V,H_0}(\lambda)\,d\lambda,$$

$\|\xi\|_1 \leq \mathrm{tr}(|V|)$, $\int_{-\infty}^{\infty} \xi(\lambda)\,d\lambda = \mathrm{tr}(V)$

Particular case: tr is finite

$$\mathrm{tr}\big[f(H_0+V) - f(H_0)\big] = \int_{-\infty}^{\infty} f(\lambda)\,d\,\mathrm{tr}\big[E_{H_0+V}(\lambda) - E_{H_0}(\lambda)\big]$$

$$= \mathrm{tr}\big[E_{H_0+V}(\lambda) - E_{H_0}(\lambda)\big]\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(\lambda)\mathrm{tr}\big[E_{H_0+V}(\lambda) - E_{H_0}(\lambda)\big]\,d\lambda$$

Spectral shift functions (SSF) II

How does the spectrum shift (relative to $\lambda \in \mathbb{R}$)?

how far:

$$\mathrm{tr}\big[VE_{H_0}(\lambda)E_{H_0+V}(\lambda)^{\perp}\big] - \mathrm{tr}\big[VE_{H_0}(\lambda)^{\perp}E_{H_0+V}(\lambda)\big] =: \gamma_{H_0,H_0+V}(\lambda)$$

Observation:

$$QP^{\perp} - Q^{\perp}P = Q(I-P) - Q^{\perp}P = Q - (Q+Q^{\perp})P = Q - P$$

$$\gamma_{H_0,H_0+V}(\lambda) = \mathrm{tr}\big[V\big(E_{H_0}(\lambda) - E_{H_0+V}(\lambda)\big)\big]$$

$$\gamma_{H_0,H_0+V}(\lambda) = \eta_{H_0,H_0+V}(\lambda) + \eta_{H_0+V,H_0}(\lambda)$$

$$\eta_{H_0,H_0+V}(\lambda) = \text{“}\mathrm{tr}\big[(H_0+V-\lambda I)\big(E_{H_0}(\lambda) - E_{H_0+V}(\lambda)\big)\big]\text{”}$$

Koplienko's SSF '84:

$$\eta_{H_0,H_0+V}(\lambda) := \mathrm{tr}[VE_{H_0}(\lambda)] + \int_{-\infty}^{\lambda} \xi_{H_0,H_0+V}(t)\,dt$$

More general perturbations

$\mathcal{A}$ is a semi-finite von Neumann algebra

τ is a normal faithful semi-finite trace on $\mathcal{A}$

$H_0 = H_0^*$ is affiliated with $\mathcal{A}$

$V \in \mathcal{L}_1(\mathcal{A}, \tau) = L_1(\mathcal{A}, \tau) \cap \mathcal{A}$ (τ-trace class)

[Carey, Pincus '77; Azamov, Dodds, Sukochev '06]:

$$\tau\big[f(H_0 + V) - f(H_0)\big] = \int_{-\infty}^{\infty} f'(\lambda)\xi_{H_0+V,H_0}(\lambda)\,d\lambda,$$

for $f \in C^1(\mathbb{R})$, $f' = \mathcal{F}(m)$, $\|m\| < \infty$; $\quad \|\xi\|_1 \leq \tau(|V|)$

[Koplienko '84] for $\tau = \mathrm{tr}$:

$$\tau\big[f(H_0 + V) - f(H_0) - Vf'(H_0)\big] = \int_{-\infty}^{\infty} f''(\lambda)\eta_{H_0,H_0+V}(\lambda)\,d\lambda$$

Monotonicity

$$f \text{ is increasing}, \ldots, V \geq 0 \Rightarrow \tau[f(H_0)] \leq \tau[f(H_0 + V)] \quad (1)$$

$$\text{or } \tau[f(H_0 + V) - f(H_0)] \geq 0 \quad (2)$$

[Petz '85](1): τ is finite

[Brown, Kosaki '90](1): $H_0 \geq 0$, f is continuous, $f(0) = 0$
Idea: $\tau[E_{H_0}((\lambda, \infty))] \leq \tau[E_{H_0+V}((\lambda, \infty))]$, $\lambda > 0$

(2): $V \in \mathcal{L}_1(\mathcal{A}, \tau)$, f satisfies Krein's trace formula

$$\tau\left[f(H_0 + V) - f(H_0)\right] = \int_{-\infty}^{\infty} f'(\lambda)\xi_{H_0+V,H_0}(\lambda)\, d\lambda$$

Proof: enough to show that $\xi_{H_0+V,H_0}(\lambda) \geq 0$ for a.e. $\lambda \in \mathbb{R}$.
[Kostrykin, Makarov, S '07; S '07]"Birman-Schwinger principle":

$$\xi_{H_0+V,H_0}(\lambda) = 1/\pi \lim_{\varepsilon \to 0^+} \tau\left[arg\left(I + V^{1/2}(H_0 - \lambda I - \mathrm{i}\varepsilon I)^{-1} V^{1/2}\right)\right] \geq 0.$$

Klein's convexity inequality

$$f \text{ is convex}, \ldots \Rightarrow \tau[f(H_0+V)-f(H_0)-Vf'(H_0)] \geq 0 \qquad (3)$$

"The graph of a convex function lies above the tangent lines."

Particular case: $\frac{d}{dt}\tau[f(H_0+tV)] = \tau[Vf'(H_0+tV)]$ is increasing

$$\tau[f(H_0+V)] \geq \tau[f(H_0)] + \frac{d}{dt}\bigg|_{t=0} \tau[f(H_0+tV)] \cdot (1-0)$$

[Petz '85](3): τ is finite

(3): $V \in \mathcal{L}_1(\mathcal{A},\tau)$, f satisfies Koplienko's trace formula

$$\tau\Big[f(H_0+V)-f(H_0)-Vf'(H_0)\Big] = \int_{-\infty}^{\infty} f''(\lambda)\eta_{H_0+V,H_0}(\lambda)\,d\lambda$$

Proof: enough to show that $\eta_{H_0+V,H_0}(\lambda) \geq 0$ for a.e. $\lambda \in \mathbb{R}$.

Remark: (3) was applied to show that $\eta_{H_0+V,H_0}(\lambda) \geq 0$ for τ finite.

Concavity I

$$f \text{ is concave,} \ldots \Rightarrow H \mapsto \tau[f(H)] \text{ is concave,} \tag{4}$$

$$\alpha\tau[f(A)] + (1-\alpha)\tau[f(B)] \le \tau\big[f\big(\alpha A + (1-\alpha)B\big)\big],$$

$$\alpha\tau[f(H_0 + V_1)] + (1-\alpha)\tau[f(H_0 + V_2)]$$

$$\le \tau\big[f\big(\alpha(H_0 + V_1) + (1-\alpha)(H_0 + V_2)\big)\big], \ \alpha \in [0,1]$$

$$\text{or } V \mapsto \tau[f(H_0 + V) - f(H_0)] \text{ is concave,}$$

$$\alpha\tau[f(H_0 + V_1) - f(H_0)] + (1-\alpha)\tau[f(H_0 + V_2) - f(H_0)]$$

$$\le \tau\big[f\big(H_0 + \alpha V_1 + (1-\alpha)V_2\big) - f(H_0)\big],$$

$$\text{or } [0,1] \ni t \mapsto \tau[f(H_0 + V(t)) - f(H_0)] \text{ is concave,} \tag{5}$$

$$\alpha\tau[f(H_0 + V(t_1)) - f(H_0)] + (1-\alpha)\tau[f(H_0 + V(t_2)) - f(H_0)]$$

$$\le \tau\big[f\big(H_0 + V(\alpha t_1 + (1-\alpha)t_2)\big) - f(H_0)\big]$$

[Petz '85](4): τ is finite

Concavity II

[Brown, Kosaki '90](4): $A, B \geq 0$, f is continuous, $f(0) = 0$

$t \mapsto V(t)$ is concave, f satisfies Krein's trace formula, $\|f'\|_\infty < \infty$

[Kostrykin '00](5): $\tau = \mathrm{tr}$, V is in the trace class

(5): $V \in \mathcal{L}_1(\mathcal{A}, \tau)$

Proof. $(V(t) = tV)$

$$\tau\big[f(H_0 + V(t)) - f(H_0)\big] = \int_{-\infty}^{\infty} f'(\lambda)\xi_{H_0+V(t),H_0}(\lambda)\, d\lambda.$$

May WLOG assume that $f' \geq 0$ (or consider $f(\lambda) + c\lambda$, $c > 0$).
$f' \geq 0$, bounded, decreasing; hence, to prove the concavity of

$$[0,1] \ni t \mapsto \int_{-\infty}^{\infty} f'(\lambda)\xi_{H_0+V(t),H_0}(\lambda)\, d\lambda,$$

it is enough to prove the concavity of $t \mapsto \int_{-\infty}^{\lambda} \xi_{H_0+V(t),H_0}(s)\, ds$, $\forall \lambda \in \mathbb{R}$.

Concavity III

The Birman-Solomyak spectral averaging formula:

$$\int_{-\infty}^{\lambda} \xi_{H_0+V(t),H_0}(s)\,ds = \int_0^t \tau\big[V'(s)E_{H_0+V(s)}(\lambda)\big]\,ds. \qquad (6)$$

It is enough to show that

$$t \mapsto \tau\big[V'(t)E_{H_0+V(t)}(\lambda)\big] \text{ is decreasing } \forall\lambda\in\mathbb{R}. \qquad (7)$$

$V(t) = tV$

$$\text{“}V \geq 0 \Rightarrow H_0 + tV \nearrow \Rightarrow E_{H_0+tV}(\lambda) \searrow \Rightarrow VE_{H_0+tV}(\lambda) \searrow\text{”}$$

[Birman, Solomyak '72] (6)&(7): $V(t) = tV$, $\tau = \mathrm{tr}$

[Azamov, Carey, Dodds, Sukochev '07] (6): $V(t) = tV$

[Simon '98; Gesztesy, Makarov, Naboko '99] (6)&(7): $\tau = \mathrm{tr}$

[Makarov, S] (6)&(7): general case

Positivity of Koplienko's SSF

(i) $\eta_{H_0,H_0+V}(\lambda) \geq 0$

(ii) $t_1 \leq t_2 \Rightarrow \tau\big[VE_{H_0+t_1V}(\lambda)\big] \geq \tau\big[VE_{H_0+t_2V}(\lambda)\big]$

Proof. τ is finite, $H_0 \in \mathcal{A}$:

(i)

$$\begin{aligned}\eta_{H_0,H_0+V}(\lambda) &= \tau\big[(H_0+V-\lambda I)\big(E_{H_0}(\lambda)-E_{H_0+V}(\lambda)\big)\big] \\ &= \tau\big[(H_0+V-\lambda I)E_{H_0+V}(\lambda)^{\perp}E_{H_0}(\lambda)\big] \\ &\quad - \tau\big[(H_0+V-\lambda I)E_{H_0+V}(\lambda)E_{H_0}(\lambda)^{\perp}\big]\end{aligned}$$

$$(H_0+V-\lambda I)E_{H_0+V}([\lambda,\infty)) \geq 0, \quad (H_0+V-\lambda I)E_{H_0+V}((-\infty,\lambda)) \leq 0.$$

(ii) $\gamma_{H_0,H_0+V}(\lambda) = \eta_{H_0,H_0+V}(\lambda) + \eta_{H_0+V,H_0}(\lambda) \geq 0$

$$\gamma_{H_0+t_1V,H_0+t_2V}(\lambda) = (t_2-t_1)\left(\tau\big[VE_{H_0+t_1V}(\lambda)\big] - \tau\big[VE_{H_0+t_2V}(\lambda)\big]\right).$$

τ is infinite, H_0 is unbounded:
do approximations to get (ii) and then (i).