NORMALIZERS OF TENSOR PRODUCTS

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NORMALIZERS

$B \subseteq M$ finite vN algebras

$\mathcal{N}(B) = \{u \in \mathcal{U}(M)\colon\ uBu^* = B\}$

$\mathcal{ON}(B) = \{u \in \mathcal{U}(M)\colon\ uBu^* \subseteq B\}$

Definition 1 (Dixmier)**.** In the context of masas, B is singular if $\mathcal{N}(B)'' = B$. B is regular (or Cartan) if $\mathcal{N}(B)'' = M$.

PROBLEM: What can be said about $\mathcal{N}(B)$ and $\mathcal{N}(B)''$ for general subalgebras B?

MASAS

Theorem 2 (Chifan)**.** *$A_i \subseteq M_i$ are masas in II_1 factors. Then*

$$\mathcal{N}(A_1 \overline{\otimes} A_2)'' = \mathcal{N}(A_1)'' \overline{\otimes} \mathcal{N}(A_2)''.$$

HOPE: each $u \in \mathcal{N}(A_1 \overline{\otimes} A_2)$ is $w(u_1 \otimes u_2)$ with $u_i \in \mathcal{N}(A_i)$, $w \in \mathcal{U}(A_1 \overline{\otimes} A_2)$.

Theorem 3 (Popa)**.** *If $A \subseteq M$ is a Cartan masa and $p, q \in A$ are projections of equal trace, then $upu^* = q$ for some $u \in \mathcal{N}(A)$.*

In $A \overline{\otimes} A \subseteq M \overline{\otimes} M$, there is a unitary normalizer so that

$$u(p \otimes 1)u^* = 1 \otimes p$$

and u cannot have the form $w(u_1 \otimes u_2)$.

IRREDUCIBLE SUBFACTORS

$$N \subseteq M, \qquad N' \cap M = \mathbb{C}1$$

Basic Construction

M has trace τ, inner product

$$\langle x, y \rangle = \tau(y^* x), \qquad x, y \in M.$$

$L^2(M)$ is the completion.

$$J\colon\ L^2(M) \to L^2(M) \quad \text{is} \quad Jx = x^*, x \in M.$$

Isometric because $\tau(xx^*) = \tau(x^*x)$

e_N is the projection onto $L^2(N) \subseteq L^2(M)$.

$$\langle M, e_N \rangle = \{M \cup e_N\}'', \quad \langle M, e_N \rangle' = JNJ.$$

Normal semifinite trace Tr on $\langle M, e_N \rangle$

$$\mathrm{Tr}(x e_N y) = \tau(xy), \qquad x, y \in M.$$

PROPERTIES

$N \subseteq M$ are finite von Neumann algebras.

(i) $x \mapsto e_N x$ and $x \mapsto x e_N$ are injective maps on M.

(ii) $x \mapsto e_N x$ is a *-isomorphism on N.

(iii) $e_N x e_N = \mathbb{E}_N(x) e_N$ for $x \in M$.

(iv) $e_N \langle M, e_N \rangle e_N = N e_N$.

(v) $M \cap \{e_N\}' = N$.

(vi) For $N_i \subseteq M_i$, $i = 1, 2$,

$$\langle M_1 \overline{\otimes} M_2, e_{N_1 \overline{\otimes} N_2} \rangle = \langle M_1, e_{N_1} \rangle \overline{\otimes} \langle M_2, e_{N_2} \rangle,$$

$$e_{N_1 \overline{\otimes} N_2} = e_{N_1} \otimes e_{N_2},$$

and the trace on $\langle M_1 \overline{\otimes} M_2, e_{N_1 \overline{\otimes} N_2} \rangle$ is $Tr_1 \otimes Tr_2$.

GENERAL STRATEGY

(1) View a problem about $N \subseteq M$ as being in the larger algebra $\langle M, e_N \rangle$.

(2) Do some calculations in $\langle M, e_N \rangle$.

(3) Push everything back to $N \subseteq M$.

Example 4. e_N commutes with N (simple). If $u \in \mathcal{ON}(N)$ (or $\mathcal{N}(N)$) then $u^* e_N u$ commutes with N. For $x \in N$,

$$u^* e_N u x = u^* e_N \underbrace{u x u^*}_{\in N} u = u^* u x u^* e_N u$$
$$= x u^* e_N u.$$

Thus $u^* e_N u \in N' \cap \langle M, e_N \rangle$ and

$$\mathrm{Tr}(u^* e_N u) = \mathrm{Tr}(e_N) = 1.$$

MAIN TECHNICAL THEOREM

Theorem 5. *Let $N \subseteq M$ be an irreducible inclusion of II_1 factors. The non-zero projections $f \in N' \cap \langle M, e_N \rangle$ which satisfy $\mathrm{Tr}(f) \leq 1$ are exactly those of the form*

$$u^* e_N u, \qquad u \in \mathcal{ON}(N).$$

Moreover, $\mathrm{Tr}(f) \geq 1$ for all non-zero projections $f \in N' \cap \langle M, e_N \rangle$, and all f with $Tr(f) = 1$ are central.

TENSOR PRODUCTS

Theorem 6. *Suppose $N_i \subseteq M_i$, $i = 1, 2$, irreducible.*

(i) *Each $u \in \mathcal{ON}(N_1 \overline{\otimes} N_2)$ has the form $w(u_1 \otimes u_2)$ for*

$$w \in \mathcal{U}(N_1 \overline{\otimes} N_2), \qquad u_i \in \mathcal{ON}(N_i).$$

(ii) *Same result with $\mathcal{N}(\cdot)$ replacing $\mathcal{ON}(\cdot)$ throughout.*

Corollary 7.

$$\mathcal{ON}(N_1 \overline{\otimes} N_2)'' = \mathcal{ON}(N_1)'' \,\overline{\otimes}\, \mathcal{ON}(N_2)'',$$
$$\mathcal{N}(N_1 \overline{\otimes} N_2)'' = \mathcal{N}(N_1)'' \,\overline{\otimes}\, \mathcal{N}(N_2)''.$$

For singular masas, this is Sinclair–SWW.

For general masas, this is Chifan.

REASON

$$N := N_1 \overline{\otimes} N_2 \subseteq M_1 \overline{\otimes} M_2 := M$$

$$\begin{aligned} u \in \mathcal{ON}(N) \Longrightarrow u^* e_N u &\in Z(N' \cap \langle M, e_N \rangle) \\ &= Z(N_1' \cap \langle M_1, e_{N_1} \rangle) \overline{\otimes} Z(N_2' \cap \langle M_2, e_{N_2} \rangle) \\ \Longrightarrow \quad u^* e_N u &= u_1^* e_{N_1} u_1 \otimes u_2^* e_{N_2} u_2. \end{aligned}$$

This implies that $u(u_1 \otimes u_2)^*$ commutes with e_N so is a unitary $w \in N$. Then

$$u = w(u_1 \otimes u_2)$$

GROUP-SUBGROUP INCLUSIONS

$H \subseteq G$ countable discrete groups. $L(H)' \cap L(G) = \mathbb{C}1$ precisely when each $g \in G \backslash \{e\}$ has infinitely many H-conjugates.

$$\mathcal{N}_G(H) = \{g \in G\colon \; gHg^{-1} = H\}$$
$$\mathcal{ON}_G(H) = \{g \in G\colon \; gHg^{-1} \subseteq H\}.$$

These can be very different. We can have

$$\mathcal{N}_G(H) \quad \text{generates} \quad L(H),$$
$$\mathcal{ON}_G(H) \quad \text{generates} \quad L(G).$$

Theorem 8. (i) *Each* $u \in \mathcal{ON}(L(H))$ *has the form* $u = wg$ *where* w *is a unitary in* $L(H)$ *and* $g \in \mathcal{ON}_G(H)$.

(ii) *Each* $u \in \mathcal{N}(L(H))$ *has the form* $u = wg$ *where* w *is a unitary in* $L(H)$ *and* $g \in \mathcal{N}_G(H)$.

Example 9. Index the generators of $\mathbb{F}_\infty$ by $\mathbb{Z}$. Define an automorphism ϕ by

$$\phi(g_i) = g_{i+1}, \qquad i \in \mathbb{Z}.$$

This gives an action of $\mathbb{Z}$ on $\mathbb{F}_\infty$.

$$G = \mathbb{F}_\infty \rtimes \mathbb{Z},$$

$$H = \langle g_i \colon \; i \geq 0 \rangle.$$

$\mathcal{N}_G(H) = H$,

$$\mathcal{ON}_G(H) \quad \text{generates} \quad G.$$

At the algebra level, we get

$$\mathcal{ON}(L(H))'' = L(G)$$

$$\mathcal{N}(L(H))'' = L(H)$$

$\mathcal{N}(B_1 \overline{\otimes} B_2)'' \neq \mathcal{N}(B_1)'' \overline{\otimes} \mathcal{N}(B_2)''$

Example 10. (M, τ) is a II_1 factor, $p \in M$ is a projection with $0 < \tau(p) < 1/2$.

$$B = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x \in pMp,\ y \in p^\perp M p^\perp \right\}.$$

Then $\mathcal{N}(B)'' = B$. Consider $B \overline{\otimes} B \subseteq M \overline{\otimes} M$. Then

$$p \otimes p^\perp \perp p^\perp \otimes p, \quad \tau \otimes \tau(p \otimes p^\perp) = \tau \otimes \tau(p^\perp \otimes p)$$

so there is a self–adjoint unitary $u \in M \overline{\otimes} M$ so that

$$u(p \otimes p^\perp)u^* = p^\perp \otimes p,$$

$$u(p \otimes p)u^* = p \otimes p,$$

$$u(p^\perp \otimes p^\perp)u^* = p^\perp \otimes p^\perp.$$

These projections lie in $Z(B \overline{\otimes} B)$ so $u \notin B \overline{\otimes} B$ and $u \in \mathcal{N}(B \overline{\otimes} B)$. Thus

$$\mathcal{N}(B)'' \overline{\otimes} \mathcal{N}(B)'' \subsetneq \mathcal{N}(B \overline{\otimes} B)''.$$

GROUPOID NORMALIZERS

$B \subseteq M$ is an inclusion of finite von Neumann algebras. A partial isometry $v \in M$ is a groupoid normalizer of B if

(i) $vBv^* \subseteq B$;

(ii) $vv^*,\ v^*v \in B$.

The set of groupoid normalizers is $\mathcal{GN}(B)$.

SIMPLE EXAMPLE

$\mathbb{M}_2$ is the 2×2 matrices,

$$\mathbb{D}_2 = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}.$$

A typical groupoid normalizer is

$$v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

where the product matrices are respectively a unitary in $\mathcal{N}(\mathbb{D}_2)$ and a projection in $\mathbb{D}_2$, as in a general theorem of Dye for masas.

Example 11. As in the Example 10,

$$B = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x \in pMp,\ y \in p^{\perp}Mp^{\perp} \right\}.$$

Partial isometries of the form

$$v = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

normalize B, so $\mathcal{GN}(B)'' = M$. Similarly $\mathcal{GN}(B \mathbin{\overline{\otimes}} B)'' = M \mathbin{\overline{\otimes}} M$ so

$$\mathcal{GN}(B \mathbin{\overline{\otimes}} B)'' = \mathcal{GN}(B)'' \mathbin{\overline{\otimes}} \mathcal{GN}(B)''.$$

NOTE: $B' \cap M \subseteq B$ holds here.

Example 12. In $\mathbb{M}_3$, the 3×3 matrices, let

$$B = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} : \alpha,\ \beta \in \mathbb{C} \right\}.$$

$$\begin{aligned} B &= \mathbb{C}I_2 \oplus \mathbb{C}I_1, \\ \mathcal{GN}(B)'' &= \mathbb{M}_2 \oplus \mathbb{M}_1, \\ B \,\overline{\otimes}\, B &= \mathbb{C}I_4 \oplus \mathbb{C}I_2 \oplus \mathbb{C}I_2 \oplus \mathbb{C}I_1, \\ \mathcal{GN}(B \,\overline{\otimes}\, B)'' &= \mathbb{M}_4 \oplus \mathbb{M}_4 \oplus \mathbb{M}_1. \end{aligned}$$

$$dim\ \mathcal{GN}(B \,\overline{\otimes}\, B)'' = 33, \quad dim\ \mathcal{GN}(B)'' \,\overline{\otimes}\, \mathcal{GN}(B)'' = 25$$

NOTE: $B' \cap \mathbb{M}_3 = \mathbb{M}_2 \oplus \mathbb{M}_1 \not\subseteq B$.

MAIN THEORM

Theorem 13. *Let $B_i \subseteq M_i$, $i = 1, 2$, be containments of finite von Neumann algebras satisfying $B_i' \cap M_i \subseteq B_i$. Then*

$$\mathcal{GN}(B_1 \overline{\otimes} B_2)'' = \mathcal{GN}(B_1)'' \overline{\otimes} \mathcal{GN}(B_2)''.$$

Moreover, each groupoid normalizer $u \in M_1 \overline{\otimes} M_2$ is approximable in $\|\cdot\|_2$–norm by sums

$$\sum_{j=1}^{k} b_j (v_{1,j} \otimes v_{2,j})$$

where $b_j \in B_1 \overline{\otimes} B_2$ and $v_{i,j} \in \mathcal{GN}(B_i)$ for $i = 1, 2$.

TOOLS

Theorem 14 (Chifan)**.** *Let B be a masa in M and let $P \in B' \cap \langle M, e_B \rangle$ be such that $P \precsim e_B$. Then there exists a groupoid normalizer v of B such that $v^* e_B v \leq P$.*

Theorem 15. *Let A be an abelian von Neumann algebra with a normal semifinite weight Φ, and let $B \subseteq M$ be finite von Neumann algebras satisfying $B' \cap M \subseteq B$. If P is a projection in $A \overline{\otimes} (B' \cap \langle M, e_B \rangle)$ such that $P \precsim 1 \otimes e_B$ in $A \overline{\otimes} \langle M, e_B \rangle$ and satisfies*

$$\Phi \otimes Tr(Pq) \leq (\Phi \otimes \tau)(q), \quad q \in A \overline{\otimes} B,$$

then $P = v^*(1 \otimes e_B)v$ for some $v \in \mathcal{G}N_{A \overline{\otimes} M}(A \overline{\otimes} B)$.

Lemma 16. *Let $B \subseteq M$ be a containment of finite von Neumann algebras and let $v \in \mathcal{G}N(B)$. Then $v^* e_B v$ is a projection in $Z(B' \cap \langle M, e_B \rangle) v^* v$.*

This connects to tensor products via

$$B' \cap \langle M, e_B \rangle = (B_1' \cap \langle M_1, e_{B_1} \rangle) \,\overline{\otimes}\, (B_2' \cap \langle M_2, e_{B_2} \rangle)$$

when $B = B_1 \,\overline{\otimes}\, B_2,\ M = M_1 \,\overline{\otimes}\, M_2$.

SAMPLE CALCULATION

$B_i \subseteq M_i$, $M = M_1 \overline{\otimes} M_2$, $B = B_1 \overline{\otimes} B_2$. $v \in \mathcal{G}N(B)$,

$$v^* e_B v \sim \sum v_j^* e_B v_j$$

where $v_j = v_{1,j} \otimes v_{2,j}$, $v_{i,j} \in \mathcal{G}N(B_i)$.

Multiply on the left by $e_B v$ to get

$$\begin{aligned} e_B v \sim \sum e_B v v_j^* e_B v_j &= \sum b_j e_B v_j \\ &= \sum e_B b_j v_j. \end{aligned}$$

Then

$$v \sim \sum b_j (v_{1,j} \otimes v_{2,j}).$$

$$Me_BM := \text{span}\{xe_By\colon x, y \in M\} \subseteq \langle M, e_B\rangle.$$

The PULL DOWN map is $\Psi\colon Me_BM \to M$,

$$\Psi(xe_By) = xy, \quad x, y \in M.$$

Unbounded, but bounded as a map

$$\Psi\colon L^1(\langle M, e_B\rangle, Tr) \to L^1(M, \tau).$$

Ψ is the pre–adjoint of the embedding $M \hookrightarrow \langle M, e_B\rangle$. so is a completely positive contraction.

Useful properties:

(i) Let $x \in L^1(\langle M, e_B\rangle)^+$. If x and $\Psi(x)$ are bounded then $\Psi(x) \geq x$.

(ii) Let $x \in L^1(\langle M, e_B\rangle)^+$. If x is unbounded, then $\Psi(x)$ is unbounded in $L^1(M)$.

$\mathcal{GN}(B)''$ *versus* $\mathcal{N}(B)''$, $\mathcal{ON}(B)''$

(1) B is a masa in M, a II_1 factor. A theorem of Dye says that each $v \in \mathcal{GN}(B)$ has the form $v = wp$ for some $w \in \mathcal{N}(B)$ and some projection $p \in B$. Thus

$$\mathcal{GN}(B)'' = \mathcal{N}(B)''.$$

(2) Let $N \subseteq M$ be an irreducible inclusion of II_1 factors. Each $v \in \mathcal{GN}(B)$ with $\tau(vv^*) \in \mathbb{Q}$ has the form $v = wp$ for some $w \in \mathcal{ON}(N)$ and some projection $p \in N$. Thus an approximation argument gives

$$\mathcal{GN}(N)'' = \mathcal{ON}(N)''.$$

BIMODULES

Projections in $N' \cap \langle M, e_N \rangle$ correspond to closed N–bimodules in $L^2(M)$ by $f \mapsto$ Range f. For $L(H) \subseteq L(G)$:

(1) Each $g \in G$ gives a double coset HgH and the closed span in $\ell^2(G)$ is an $L(H)$–bimodule.

(2) Each $g \in G$ gives a left coset gH and the closed span in $\ell^2(G)$ is a right $L(H)$–module. The range projection is $g e_{L(H)} g^{-1}$, trace 1.

(3) If $f \in L(H)' \cap \langle L(G), e_{L(H)} \rangle$ is a projection and $\mathrm{Tr}(f) < \infty$, then Range f is a finite sum of double cosets and also a finite sum of left cosets. Such projections generate an abelian algebra whose projections all have integer trace.

EXAMPLE (Izumi–Longo–Popa)

$$G = \mathbb{F}_3 = \langle a, b, c \rangle \qquad H = \mathbb{F}_2 = \langle a, b \rangle$$

Each projection $p \in W^*(c)$ generates an $L(H)$–bimodule $X_p \subseteq \ell^2(G)$ and

$$X_{p_1} \subseteq X_{p_2} \iff p_1 \leq p_2.$$

Here $L(H)' \cap \langle L(G), e_{L(H)} \rangle$ decomposes as

$$\mathbb{C}e_{L(H)} \oplus (\mathrm{II}_\infty \text{ factor})$$

Tr is infinite on the second summand.