Operator space structure of JC^*-triples

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GPOTS 21/June/2008

JC^*-triples and TROs

- on $\mathcal{B}(H)$, we define the *Jordan triple product* $\{x_1, x_2, x_3\} = (x_1 x_2^* x_3 + x_3 x_2^* x_1)/2$
  A JC^*-triple is $X \subset \mathcal{B}(H)$ (closed) subspace, closed under $\{\cdot, \cdot, \cdot\}$.

- *Ternary product* on $\mathcal{B}(H)$ is $[x_1, x_2, x_3] = x_1 x_2^* x_3$
  A TRO is $T \subset \mathcal{B}(H)$ (closed) subspace, closed under $[\cdot, \cdot, \cdot]$.

Examples

(i) $A \subset \mathcal{B}(H)$ a C^*-algebra $\Rightarrow A$ a TRO $\Rightarrow A$ a JC^*-triple

(ii) T a finite dimensional TRO $\Rightarrow T$ a direct sum of rectangular matrix spaces

(*e.g.* $M_{5,7}(\mathbb{C})$, $M_{8,1}(\mathbb{C})$).

(iii) $X = \{x \in M_n(\mathbb{C}) : x^t = x\}$ is JC^*-triple (not TRO).

Note: $X =$ symmetric $n \times n$ matrices.

Background motivation — JC^*-triples

Consider JC^*-(sub)triples $X \subset \mathcal{B}(H)$ with operator norm. $B_X = \{x \in X : \|x\| < 1\}$ unit ball

$\mathrm{Aut}(B_X)$ transitive (contains a Möbius like group).

X Banach (over $\mathbb{C}$), $\mathrm{Aut}(B_X)$ transitive $\iff$ X has a JB^*-triple product. (W. Kaup 1983). B_X a bounded symmetric domain. (Original Lie-theoretic approach by Cartan (1932) for $\dim X < \infty$. Jordan approach due to M. Koecher (1969).)

'Most' JB^*-triples are (isometric to) JC^*-triples.

Notions of equivalence?

Biholomorphic equivalence of bounded symmetric domains

Linear isometry of (unit balls of) JB^*-triples

Algebraic (JB^*-triple) isomorphism

Conclusion: We treat two JC^*-triples as the 'same' if they are (linearly) isometric

Background motivation — TROs

Associative product $[x_1, x_2, x_3] = x_1 x_2^* x_3$

Characterised by Zettl (1983) in abstract algebraic terms. Left C^*-algebra of T is $\mathscr{L}_T$ (generated by $x_1 x_2^*$). Right C^*-algebra of T is $\mathscr{R}_T$ (generated by $x_1^* x_2$). Algebraic TRO morphisms extend to left and right C^*-algebras.

Isometry of TROs does not imply algebraic TRO isomorphism. Simplest examples: $M_{2,1}(\mathbb{C})$ and $M_{1,2}(\mathbb{C})$.

Language of operator spaces gives another viewpoint.

TROs in operator space theory

- TROs T_1 and T_2 are the same $\iff$ completely isometric.

- TRO structure $\Rightarrow$ unique compatible operator space structure

- (Hamana theory) every operator space can be canonically embedded in its injective envelope and that envelope is a TRO.

Is a JC^*-triple an operator space?

Answer: Not so obviously (or canonically)
Too obvious: $X \subset \mathcal{B}(H)$ closed under triple product $\Rightarrow$ X an operator space.

The snag is that two JC^*-triples are considered the 'same' if they are isometric and this does not imply complete isometry. [Neal & Russo]

e.g. the TROs $M_{2,1}(\mathbb{C})$ and $M_{1,2}(\mathbb{C})$ are the same as JC^*-triples, but not as operator spaces. Isometric but not completely isometric.

Universal TRO

Theorem: Given a JC^*-triple X, there is a unique universal embedding $\alpha_X\colon X \to T^*(X)$ into TRO s.t.

- α_X is a JC^*-triple isom onto its range
- $T^*(X) =$ (closure of) subTRO generated by $\alpha_X(X)$
- $\forall\ \pi\colon X \to T$ JC^*-triple homom into TRO T, $\exists!$ TRO-hom $\tilde{\pi}\colon T^*(X) \to T$ lifting π $(\tilde{\pi}(\alpha_X(x)) = \pi(x) \forall x \in X)$

$$\begin{array}{ccc} T^*(X) & & \\ \alpha_X \uparrow & \overset{\tilde{\pi}}{\searrow} & \\ X & \xrightarrow[\pi]{} & T \end{array}$$

Consequence: Given a JC^*-triple X (up to equivalence), for any realisation $X \subset \mathcal{B}(H)$ as a JC^*-subtriple, TRO$(X) \cong T^*(X)/\mathcal{I}$ for $\mathcal{I} \subset T^*(X)$ a TRO ideal with $\mathcal{I} \cap \alpha_X(X) = \{0\}$.
Each such $\mathcal{I}$ yields an operator space structure $X_{\mathcal{I}}$ on X — all possible JC^*-operator space structures

How to find $T^*(X)$?

Tripotents are a key concept in JC^*-triples X: $e \in X$ with $\{e, e, e\} = e$. Associated Peirce spaces $X_\lambda(e) = \{x \in X : 2\{e, e, x\} = \lambda x\}$
$X_\lambda(e)$ nonzero implies $\lambda \in \{0, 1, 2\}$.
$X_2(e)$ is a Jordan $*$-algebra: Jordan product $x \cdot y = \{x, e, y\}$, conjugation $x \mapsto \{e, x, e\}$

A tripotent $e \in X$ is called unitary if $X_2(e) = X$.
Propostion: If $\exists$ unitary tripotent $e \in X$, $T^*(X)$ 'equals' the universal C^*-algebra generated by the JC^*-algebra structure. (computed in 1970s – by

Alfsen, Stormer, Hanche-Olsen).

- $X = M_n$, $T^*(M_n) = M_n \oplus M_n$, $\alpha_X(x) = x \oplus x^t$ $(n \geq 2)$

- $X = \mathcal{B}(H)$, $T^*(X) = \mathcal{B}(H) \oplus \mathcal{B}(H)$, $\alpha_X(x) = x \oplus x^t$

  TRO ideals with $\mathcal{I} \cap \alpha_X(X) = \{0\}$ are $\mathcal{I} = \{0\}$, (always gives $\mathrm{MAX}_{JC}(X)$), $\mathcal{I} = \{0\} \oplus \mathcal{B}\,(H)$, $\mathcal{I} = \mathcal{B}(H) \oplus \{0\}$, $\mathcal{I} = \{0\} \oplus \mathcal{K}\,(H)$, $\mathcal{I} = \mathcal{K}(H) \oplus \{0\}$, . . .

- $X = \{x \in \mathcal{B}(H) : x^t = x\}$ $(\dim \geq 2)$ $T^*(X) = \mathcal{B}(H)$

- $X = \{x \in \mathcal{B}(H) : x^t = -x\}$ ($6 \leq \dim H$, $\dim H$ even or ∞) $T^*(X) = \mathcal{B}(H)$, $\alpha_X(x) = x$.

- $X = V_k =$ spin factor ($k \geq 3$) ($\dim V_k = k + 1$) $T^*(V_{2n}) = M_{2^n}$, $T^*(V_{2n+1}) = M_{2^n} \oplus M_{2^n}$, $T^*(V_k) =$ CAR if k infinite.

For the other Cartan factors (building blocks for JC^*-triples)

- $X = M_{k,n} = \mathcal{B}(H, K)$, $T^*(X) = \mathcal{B}(H, K) \oplus \mathcal{B}(K, H)$, $\alpha_X(x) = x \oplus x^t$ ($2 \leq n < k$)

- $X = \{x \in M_{2n+1} : x^t = -x\}$ $(n \geq 2)$, $T^*(X) = M_{2n+1}$, $\alpha_X(x) = x$.

- $X = \ell_k^2 = M_{k,1}$, $T^*(X) = \bigoplus_{j=1}^k \mathcal{B}\left(\Lambda^j(X), \Lambda^{j-1}(X)\right)$ contained in Fock space on dimension k ($\alpha_X(X) =$ span of k annihilation operators for o.n basis of ℓ_k^2, operators satisfying CAR)

- $X = H$ general Hilbert space, $T^*(X) =$ TRO generated by embedding in Fock space.

Further results

T^* is an exact functor JC^*-triples $\to$ TROs

$\dim X < \infty$ (X a JC^*-triple) $\Rightarrow \dim T^*(X) < \infty$

Corollary (results of Neal & Russo) Characterisation of (finite dimensional) JC^*-subtriples $X \subset \mathcal{B}(H)$ up to complete isometry.

Interesting examples (say for injective envelopes)?