

Equiangular tight frames from complex Seidel matrices containing cube roots of unity

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A Parseval frame consists of a set of vectors $\{f_1, \dots, f_n\} \subseteq \mathbb{C}^k$ satisfying

$$\sum_{i=1}^n |\langle x, f_i \rangle|^2 = \|x\|^2 \quad \text{for all } x \in \mathbb{C}^k.$$

(We'll simply call this an (n, k) -frame.)

FACT: A set $\{f_i\}_{i=1}^n$ is an (n, k) -frame if and only if

$$x = \sum_{i=1}^n \langle x, f_i \rangle f_i \quad \text{for all } x \in \mathbb{C}^k.$$

This gives a way to reconstruct a vector, and is useful in:

- Signal Processing
- Data Transmission
- Quantum Computing

Definition. An (n, k) -frame $\{f_1, \dots, f_n\}$ is called uniform if there is a constant $u > 0$ such that

$$\|f_i\| = u \quad \text{for all } i.$$

Definition. We say an (n, k) -frame is equiangular if there exists a constant c such that

$$|\langle f_i/\|f_i\|, f_j/\|f_j\| \rangle| = c \quad \text{for all } i \neq j.$$

Equiangular \implies Uniform.

Uniform frames are easy to construct. Equiangular frames are much harder to find.

Equiangular frames contain a lot of structure, and are important in many pure and applied problems. Also, Engineers care because:

- Uniform frames are optimal for 1 erasure.
- Equiangular frames are optimal for 1 or 2 erasures.

Analysis Operator and Seidel Matrix

A frame $\{f_1, \dots, f_n\} \subseteq \mathbb{C}^k$ gives rise to an isometric embedding $V : \mathbb{C}^k \rightarrow \mathbb{C}^n$ given by

$$(Vx)_j := \langle x, f_j \rangle \text{ for } j \in \{1, 2, \dots, n\}.$$

We call V the Analysis Operator, and V^* has matrix $(\vec{f}_1 \ \dots \ \vec{f}_n)$.

Also, $V^*V = I_k$ and the projection $VV^* = (\langle f_i, f_j \rangle)_{i,j}$ is the Grammian matrix.

Conversely, if P is an $n \times n$ selfadjoint projection we can factor it as $P = VV^*$ for an $n \times k$ matrix V , and the columns of V^* are an (n, k) -frame.

If our frame is equiangular, with $|\langle f_i, f_j \rangle| = c_{n,k}$ for all i, j , we may write

$$Q := \frac{1}{c_{n,k}} \left(VV^* - \frac{k}{n} I_n \right)$$

and we call Q a Seidel Matrix.

Fact: Q is a selfadjoint $n \times n$ matrix satisfying $Q_{ii} = 0$ for all i and $|Q_{ij}| = 1$ for all $i \neq j$.

Theorem (Holmes and Paulsen). *Let Q be a selfadjoint $n \times n$ matrix Q with $Q_{ii} = 0$ and $|Q_{ij}| = 1$ for all $i \neq j$. Then the following are equivalent:*

- (a) Q is the Seidel matrix of an equiangular (n, k) -frame for some k ;*
- (b) $Q^2 = \mu Q + (n - 1)I$ for some necessarily real number μ ; and*
- (c) Q has exactly two eigenvalues $\lambda_1 < \lambda_2$.*

Note: Condition (b) is particularly useful since it is easy to check for a given Q .

From Real to Complex

If the entries of Q are real, then $Q_{i,j} \in \{-1, 1\}$ for $i \neq j$, and Q is the adjacency matrix of an (undirected) graph on n vertices.

There is a well-developed theory for these matrices, and it's known there is a one-to-one correspondence between equiangular (n, k) -frames and objects called regular 2-graphs.

It's natural to try to extend this to the case when the off-diagonal entries of Q are m^{th} roots of unit.

We'll consider the case when $m = 3$ and the off-diagonal entries are cube roots of unity:

$$\{1, \omega, \omega^2\}.$$

Recall: $\bar{\omega} = \omega^2$.

We seek *cube root Seidel matrices* of equian-
gular frames; that is, a selfadjoint $n \times n$ matrix
 Q with $Q_{ii} = 0$ and $Q_{ij} \in \{1, \omega, \omega^2\}$ for all $i \neq j$,
which satisfies

$$Q^2 = \mu Q + (n - 1)I$$

for some necessarily real number μ .

We'll assume at least one $Q_{ij} \neq 1$, since oth-
erwise this is a trivial case.

Equivalence of Frames

We say that two (n, k) -frames $\{f_1, f_2, \dots, f_n\}$ and $\{g_1, g_2, \dots, g_n\}$ in \mathbb{C}^k are *unitarily equivalent* if there exists a unitary U on \mathbb{C}^k such that for all $i \in \{1, 2, \dots, n\}$, $g_i = U f_i$.

We say that they are *switching equivalent* if there exist a unitary U on \mathbb{C}^k , a permutation π on $\{1, 2, \dots, n\}$ and a family of unimodular constants $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ such that for all $i \in \{1, 2, \dots, n\}$, we have $g_i = \lambda_i U f_{\pi(i)}$.

Two (n, k) -frames are switching equivalent if and only if their Seidel matrices can be obtained from each other by conjugating with a diagonal unitary and a permutation matrix.

If Q' is an $n \times n$ cube root Seidel matrix, then it is switching equivalent to a cube root Seidel matrix of the form

$$Q = \begin{pmatrix} 0 & 1 & \cdots & \cdots & 1 \\ 1 & 0 & * & \cdots & * \\ \vdots & * & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & * \\ 1 & * & \cdots & * & 0 \end{pmatrix} \quad (1)$$

Moreover, the frame for Q' is equiangular if and only if the frame for Q is equiangular.

Why? Given Q' let U be the unitary diagonal matrix

$$U := \begin{pmatrix} 1 & & & & \\ & Q'_{12} & & & \\ & & Q'_{13} & & \\ & & & \cdots & \\ & & & & Q'_{1n} \end{pmatrix}$$

and set $Q := UQ'U^*$. We then see that $(Q')^2 = \mu Q' + (n-1)I$ if and only if $Q^2 = \mu Q + (n-1)I$

Terminology: We call (1) the *standard form*.

We wish to narrow our search to possible values of (n, k) .

For an $n \times n$ cube root Seidel matrix, we create a directed graph with vertices $\{1, \dots, n\}$ and

- if $Q_{i,j} = 1$ we draw no edges between i and j ,
- if $Q_{i,j} = \omega$ we draw one edge from i to j , and
- if $Q_{i,j} = \omega^2$ we draw one edge from j to i .

Example:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & \omega & \omega^2 & 1 \\ 1 & \omega^2 & 0 & \omega & \omega \\ 1 & \omega & \omega^2 & 0 & 1 \\ 1 & 1 & \omega^2 & 1 & 0 \end{pmatrix}$$

Recall: Q is selfadjoint so $Q_{i,j} = \overline{Q_{j,i}}$.

Note: Our directed graph has at most one edge between vertices, may have cycles, and has no loops.

Given a cube root Seidel matrix in standard form, we let

$$e := \frac{n - \mu - 2}{3}.$$

If Q satisfies $Q^2 = \mu Q + (n - 1)I$ we can show:

- e is an integer
- For each row of Q other than the first,
 - there are e entries equal to ω
 - there are e entries equal to ω^2
 - there are $e + \mu + 1$ entries equal to 1.

In terms of the graph, this means that for any vertex (other than the isolated vertex 1), the out-valency is e , the in-valency is e , and there are $e + \mu + 1$ vertices not connected by an edge.

Also, $Q^2 = \mu Q + (n - 1)I$ is a statement about paths of length 2.

So for a cube root Seidel matrix in standard form, we choose $i, j \geq 2$ with $Q_{i,j} = \omega$ and let

$$\begin{aligned}\alpha &= \#\{k : Q_{ik} = \omega \text{ and } Q_{kj} = \omega^2\} \\ \beta &= \#\{k : Q_{ik} = \omega \text{ and } Q_{kj} = \omega\} \\ \gamma &= \#\{k : Q_{ik} = \omega \text{ and } Q_{kj} = 1\} \\ a &= \#\{k : Q_{ik} = \omega^2 \text{ and } Q_{kj} = \omega^2\} \\ b &= \#\{k : Q_{ik} = \omega^2 \text{ and } Q_{kj} = \omega\} \\ c &= \#\{k : Q_{ik} = \omega^2 \text{ and } Q_{kj} = 1\} \\ A &= \#\{k : Q_{ik} = 1 \text{ and } Q_{kj} = \omega^2\} \\ B &= \#\{k : Q_{ik} = 1 \text{ and } Q_{kj} = \omega\} \\ C &= \#\{k : Q_{ik} = 1 \text{ and } Q_{kj} = 1\}.\end{aligned}$$

Note: All these variables depend on i and j .

Then Q satisfies $Q^2 = \mu Q + (n - 1)I$ (and corresponds to an equiangular frame) implies that following equations are satisfied:

$$\alpha - B = -\frac{2\mu}{3} - \frac{1}{3} \quad (\text{Eq. 1})$$

$$\beta - C = -\frac{2\mu}{3} - \frac{4}{3} \quad (\text{Eq. 2})$$

$$\gamma + B + C = \frac{n}{3} + \mu \quad (\text{Eq. 3})$$

$$a - C = -\frac{\mu}{3} - \frac{2}{3} \quad (\text{Eq. 4})$$

$$b + B + C = \frac{n}{3} + \frac{\mu}{3} - \frac{1}{3} \quad (\text{Eq. 5})$$

$$c - B = -\frac{\mu}{3} + \frac{1}{3} \quad (\text{Eq. 6})$$

$$A + B + C = \frac{n}{3} + \frac{2\mu}{3} + \frac{1}{3} \quad (\text{Eq. 7})$$

for all $i \neq j$ with $Q_{i,j} = \omega$.

Note: We have 7 equations in 9 unknown with B and C as the two free variables.

Since all above variables must be integers, this gives many necessary conditions . . .

Let Q be a cube root Seidel matrix satisfying $Q^2 = \mu Q + (n - 1)I$.

Proposition 1. *If $e := \frac{n-\mu-2}{3}$, then e is an integer with*

$$e \equiv 0 \pmod{3}$$

and

$$\frac{2n}{9} \leq e \leq \frac{4n-9}{9}.$$

Proposition 2. *The following hold:*

- (a) *The value μ is an integer and $\mu \equiv 1 \pmod{3}$.*
- (b) *The integer n satisfies $n \equiv 0 \pmod{3}$.*
- (c) *If $\lambda_1 < \lambda_2$ are the eigenvalues of Q , then λ_1 and λ_2 are integers with $\lambda_1 \equiv 2 \pmod{3}$ and $\lambda_2 \equiv 2 \pmod{3}$.*
- (d) *$k = \frac{n}{2} - \frac{\mu n}{2\sqrt{4(n-1)+\mu^2}}$, and also the integer $4(n-1) + \mu^2$ is a perfect square with $4(n-1) + \mu^2 \equiv 0 \pmod{9}$.*

Algorithm for possible (n, k) values

Begin with a value of n that is divisible by 3. (See Proposition 2(b).)

Step 1: Find all values of e satisfying $\frac{2n}{9} \leq e \leq \frac{4n-9}{9}$ with $e \equiv 1 \pmod{3}$. (See Proposition 1.)

Step 2: For each e from Step 1, calculate the value of $\mu = n - 3e - 2$.

Step 3: For each μ from Step 2, calculate the value of $k = \frac{n}{2} - \frac{\mu n}{2\sqrt{4(n-1)+\mu^2}}$ (see Proposition 2(d)).

The only allowable equiangular (n, k) -frames with cube root Seidel matrices are those with k equal to an integer greater than 1 and less than n .

Table 1. All possible (n, k) values for cube root Seidel matrices of equiangular frames with $k < n \leq 100$.

n	k	μ	e	λ_1	λ_2
9	6	-2	3	-7	5
33	11	4	9	-4	8
36	21	-2	12	-7	5
45	12	7	12	-4	11
51	34	-5	18	-10	5
81	45	-2	27	-10	8
96	76	-14	36	-19	5
99	33	7	30	-7	14

We started with the smallest case $n = 9$ and $k = 6$. Using a computer search we found a solution.

$$Q = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega & \omega & \omega^2 & \omega^2 & \omega^2 \\ 1 & 1 & 0 & \omega^2 & \omega^2 & \omega^2 & \omega & \omega & \omega \\ 1 & \omega^2 & \omega & 0 & \omega & \omega^2 & 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega & \omega^2 & 0 & \omega & \omega & \omega^2 & 1 \\ 1 & \omega^2 & \omega & \omega & \omega^2 & 0 & \omega^2 & 1 & \omega \\ 1 & \omega & \omega^2 & 1 & \omega^2 & \omega & 0 & \omega^2 & \omega \\ 1 & \omega & \omega^2 & \omega^2 & \omega & 1 & \omega & 0 & \omega^2 \\ 1 & \omega & \omega^2 & \omega & 1 & \omega^2 & \omega^2 & \omega & 0 \end{pmatrix}$$

This gives a previously unknown equiangular frame.

Afterward, we used the fact that the associated graph would be (ignoring the isolated vertex) an $(8,3)$ regular directed graph, and we were able to obtain this solution from graph theory results and prove that it is unique up to switching equivalence.

In general, a cube root Seidel matrix for an equiangular (n, k) -frame gives an $(n - 1, e)$ regular directed graph.

The next allowed parameters in our table are $(n, k) = (33, 11)$ which requires an $(n - 1, e) = (32, 9)$ regular directed graph. Graph theorists do not know if such a thing exists.

Likewise, for larger values in our table nothing is known about the existence of regular directed graphs.

Also, for matrices of these larger sizes computer searches are too slow.

Another technique: Bootstrapping . . .

Proposition. *Suppose that Q_1 is an $n_1 \times n_1$ Seidel matrix and Q_2 is an $n_2 \times n_2$ Seidel matrix satisfying the equation*

$$Q_i^2 = -2Q_i + (n_i - 1)I_{n_i} \quad \text{for } i \in \{1, 2\}.$$

Then the matrix

$$Q := (Q_1 + I_{n_1}) \otimes (Q_2 + I_{n_2}) - I_{n_1 n_2}$$

is an $n_1 n_2 \times n_1 n_2$ Seidel matrix satisfying the equation

$$Q^2 = -2Q + (n_1 n_2 - 1)I_{n_1 n_2}.$$

Theorem. For each $m \in \mathbb{N}$ there exists a non-trivial $9^m \times 9^m$ cube root Seidel matrix Q satisfying $Q^2 = -2Q + (9^m - 1)I_{9^m}$.

Proof: When $m = 1$ we showed earlier that there exists a 9×9 cube root Seidel matrix Q' satisfying $(Q')^2 = -2Q' + 8I_9$.

To obtain matrices of size $9^m \times 9^m$ we iterate the construction of the preceding proposition: We form

$$Q := \underbrace{[(Q' + I_9) \otimes \dots \otimes (Q' + I_9)]}_{m \text{ times}} - I_{9^m}.$$

□

Corollary. There exist cube root Seidel matrices of equiangular (n, k) -frames for arbitrarily large n .

Table 2. All possible (n, k) values for cube root Seidel matrices of equiangular frames with $k < n \leq 100$.

n	k	μ	e	λ_1	λ_2	Do they exist?
9	6	-2	3	-7	5	Yes. (Stated Earlier)
33	11	4	9	-4	8	Unknown.
36	21	-2	12	-7	5	Unknown.
45	12	7	12	-4	11	Unknown.
51	34	-5	18	-10	5	Unknown.
81	45	-2	27	-10	8	Yes. (Bootstrap)
96	76	-14	36	-19	5	Unknown.
99	33	7	30	-7	14	Unknown.

Preprint Available

Equiangular tight frames from complex Seidel matrices containing cube roots of unity,
by Bernhard Bodmann, Vern I. Paulsen, and
Mark Tomforde.

<http://arxiv.org/abs/0805.2014>.