Strong Singularity Constants for Subfactors of II_1 Factors

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- B a unital von Neumann subalgebra of M, $\mathbb{E}_B$ the unique normal τ-preserving conditional expectation from M onto B.
- e_B is the orthgonal projection from $L^2(M)$ onto $L^2(B)$, implementing $\mathbb{E}_B$, called the Jones projection.

Singularity

Definition

(Dixmier) B is singular in M if every unitary $u \in M$ with $uBu^* = B$ must be in B.

Examples

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2. $M = R \rtimes S_n$ where R is the hyperfinite II_1 factor and S_n ($n \geq 3$) is the symmetric group on n elements, $B = R \rtimes S_{n-1}$ where S_{n-1} is considered as any subgroup that fixes a single element.

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3. Any subfactor of index strictly between 3 and 4.
4. M any II_1 factor, p a projection with $\tau(p) \neq 1/2$, $B = pMp + p^\perp Mp^\perp$.

α-Strong Singularity

Definition

(Sinclair & Smith) B is α-strongly singular in M if there exists $0 < \alpha \leq 1$ such that for all unitaries $u \in M$,

$$\alpha \| u - \mathbb{E}_B(u) \|_2 \leq \| \mathbb{E}_B - \mathbb{E}_{uBu^*} \|_{\infty,2}$$

If $\alpha = 1$, say that B is strongly singular in M. If B is α-strongly singular for some α, must it always be the case that α can be improved to one?

The WAHP

Definition

(Robertson, Sinclair, & Smith) B has the weak asymptotic homomorphism property (WAHP) if for every $\varepsilon > 0$ and $x_1, \ldots, x_n$, $y_1, \ldots, y_n \in M$, there exists a unitary $u \in B$ such that

$$\|\mathbb{E}_B(x_i u y_j) - \mathbb{E}_B(x_i) u \mathbb{E}_B(y_j)\|_2 < \varepsilon$$

for all $1 \leq i \leq n$, $1 \leq j \leq m$.

Answers for Masas

Theorem

(Sinclair, Smith, White, W) Every singular masa in M has the WAHP, and so is strongly singular.

Corollary

The tensor product of singular masas is again a singular masa.

Could the equivalence between WAHP and singularity hold in the subfactor case?

Subfactors

Definition

If $N \subseteq M$ is a unital inclusion of II_1 factors, N has finite Jones Index $[M : N]$ in M if the algebra $(M \cup e_N)'' := \langle M, e_N \rangle$ is finite. If not, N will be said to have infinite index.

The algebra $\langle M, e_N \rangle$ will be a factor and so is endowed with a unique normal, faithful, tracial state. If it is finite, define the index to be the reciprocal of the trace of e_N.

Theorem

(Jones) The index takes values in the set $\{4\cos^2(\frac{\pi}{n}) : n \geq 3\} \cup [4, \infty]$.

Big open question: can every index value be attained by N a subfactor of the hyperfinite II_1 factor $\mathcal{R}$ if in addition we assume that $N' \cap \mathcal{R} = \mathbb{C}$? All singular subfactors will have this property...

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- counterexample uses a Pimsner-Popa basis for M over N.
- For a crossed product of a II_1 factor N by a finite group G, a basis for $M = N \rtimes G$ over N is given by the image of the group elements.

A Global Constant

Theorem

Every singular subfactor is $\frac{1}{13}$-strongly singular.

Proof uses a result of Popa, Sinclair, and Smith and the fact that every groupoid normalizer of a subfactor is the product of a unitary normalizer times a projection.

Theorem

If $N \subseteq M$ is a finite-index singular subfactor and $N' \cap \langle M, e_N \rangle$ is 2-dimensional, then N is $\sqrt{\frac{[M:N]-2}{[M:N]-1}}$-strongly singular. If the index is infinite and the same relative commutant hypothesis holds, then N is strongly singular.

The subfactor examples mentioned at the beginning of this talk all satisfy the relative commutant assumption, so there are "many" such objects.

An Interesting Example

From Grossman/Jones: quadrilateral of GHJ subfactors associated to D_5 at the trivalent vertex. The intermediate subfactors P' and Q' in dual quadrilateral are unitarily conjugate and have index $2+\sqrt{2}$, so singular.

Observation of Grossman: the unitary is expectation zero onto either P' or Q', but

$$\|\mathbb{E}_{P'} - \mathbb{E}_{Q'}\|_{\infty,2} \leq \|e_{P'} - e_{Q'}\|_2$$

and computations have (hopefully!) shown that the right-hand side is not one!

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2. Can any nontrivial finite index inclusion have $\alpha = 1$?
3. Can any infinite index inclusion have $\alpha \neq 1$?
4. Does every singular masa in a II_1 factor have the asymptotic homomorphism property?