Renault's Equivalence Theorem for Groupoid Dynamical Systems and Beyond

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Ideal Structure

- I want to study the fine structure or ideal structure of crossed product C^*-algebras formed from groupoid dynamical systems.
- These are $25 words for trying to understand the structure of these algebras via their representation theory and their primitive ideal spaces in particular.
- Much of this talk is expository and serves as an advertisement for a paper Paul Muhly and I recently published in the *NYJM Monographs*.
- The "and Beyond" in the title was based on my hopelessly optimistic plan to mention some applications to work in progress on the Brauer semigroup of a groupoid (work with Jon Brown and Paul Muhly), and to a project involving the corresponding equivalence theorem for Fell bundles (work with Paul Muhly).

Imprimitivity Theorems

- One of the main tools for studying the ideal structure of any crossed product are Morita equivalence results often described as imprimitivity theorems.

Theorem (Mackey's Imprimitivity Theorem)

Suppose that G is a locally compact group and that H is a closed subgroup. Then $C^*(H)$ is Morita equivalent to the crossed product $C_0(G/H) \rtimes_{\mathrm{lt}} G$ via an imprimitivity bimodule X built out of $C_c(G)$.

This result gives us an equivalence between the representations of $C^*(H)$ and $C_0(G/H) \rtimes_{\mathrm{lt}} G$: if L is a representation of $C^*(H)$, then $\mathsf{X}\text{–}\mathrm{Ind}\, L$ is the corresponding representation of $C_0(G/H) \rtimes_{\mathrm{lt}} G$. There is a natural map $\Phi : C^*(G) \to M\big(C_0(G/H) \rtimes_{\mathrm{lt}} G\big)$. Then $\mathrm{Ind}_H^G L = \mathsf{X}\text{–}\mathrm{Ind}\, L \circ \Phi$.

Groupoid Actions

- Throughout, I'll assume that all my groupoids are second countable, locally compact, and for this talk, Hausdorff.
- We also assume that our groupoids are equipped with a continuous Haar system $\lambda = \{ \lambda^u \}_{u \in G^{(0)}}$. This allows us to form the associated groupoid C^*-algebra $C^*(G) = C^*(G, \lambda)$.
- In order for a groupoid G to act on X, X must be fibred over $G^{(0)}$ (thus we need a "structure map" $r : X \to G^{(0)}$).
- Then we require that there be a continuous map $(\gamma, x) \mapsto \gamma \cdot x$ from $G * X := \{ (\gamma, x) : s(\gamma) = r(x) \}$ to X with the usual properties: e.g., $\eta \cdot (\gamma \cdot x) = (\eta\gamma) \cdot x$ when $(\eta, \gamma) \in G^{(2)}$.
- The action is free if $\gamma \cdot x = x$ implies $\gamma = r(x)$, and proper if $(\gamma, x) \mapsto (\gamma \cdot x, x)$ is a proper map from $G * X \to X \times X$.

Equivalent Groupoids

Definition

Two groupoids G and H are equivalent if there is a locally compact space Z which is both a free and proper left G-space and a free and proper right H-space such that the actions commute and

1. r_Z induces a homeomorphism of Z/H onto $G^{(0)}$, and
2. s_Z induces a homeomorphism of $G\backslash Z$ onto $H^{(0)}$.

We call Z a (G, H)–equivalence

Example

A groupoid is transitive if given $u, v \in G^{(0)}$, there is a $\gamma \in G$ such that $r(\gamma) = u$ and $s(\gamma) = v$. If G is transitive, let $H := G_u^u = \{\, \gamma \in G : r(\gamma) = u = s(\gamma) \,\}$ be the stability group at u. Then $G^u = r^{-1}(u)$ is a free and proper left G-space and a free and proper right H-space. Since G is second countable, $s|_{G^u}$ must be open (Ramsay). It follows that G^u is a (G, H)–equivalence.

Renault's Equivalence Theorem for Groupoids

Theorem (Equivalence Theorem for Groupoids, [MRW, '87])

Suppose that G and H are equivalent via Z. Then we can build a $C^*(G,\lambda_G)$ – $C^*(H,\lambda_H)$-imprimitivity bimodule out of $C_c(Z)$. In particular, $C^*(G)$ and $C^*(H)$ are Morita equivalent.

Example

Suppose that G is a transitive groupoid and that $H = G_u^u$ for $u \in G^{(0)}$. Then $C^*(G)$ and $C^*(H)$ are Morita equivalent.

Example

If G is a group and H is a closed subgroup, then the transformation group groupoid $\mathscr{G} = (G, G/H)$ is a transitive groupoid with stability group H. Hence we recover Mackey's Imprimitivity Theorem.

Renault's Equivalence Theorem for Groupoids

Theorem (Equivalence Theorem for Groupoids, [MRW, '87])

Suppose that G and H are equivalent via Z. Then we can build a $C^*(G, \lambda_G) - C^*(H, \lambda_H)$-imprimitivity bimodule out of $C_c(Z)$. In particular, $C^*(G)$ and $C^*(H)$ are Morita equivalent.

Example

Let λ and β be continuous Haar systems on G. Since G is a (G, G)-equivalence, $C^*(G, \lambda)$ and $C^*(G, \beta)$ are Morita equivalent.

$C_0(X)$-algebras and Upper Semicontinuous Bundles

- We want use our frequent theorem points to upgrade to dynamical systems. But recall that groupoids act on fibred objects.
- The best way to accomplish this is to require that A be a $C_0(G^{(0)})$-algebra. Then $A = \Gamma_0(\mathscr{A})$ is the section algebra of an upper semicontinuous C^*-bundle $p : \mathscr{A} \to G^{(0)}$.
- Thus $\mathscr{A}$ is a topological space and $p : \mathscr{A} \to G^{(0)}$ is a continuous open surjection such that each fibre $A(u) := \mathscr{A}_u = p^{-1}(u)$ is a C^*-algebra and such that $a \mapsto \|a\|$ is upper semicontinuous on $\mathscr{A}$. (Plus a number of technical conditions that ensure, for example, that the sections form an algebra.)
- That $C_0(G^{(0)})$-algebras are the sections of such bundles goes back to work of Hofmann and Dupré & Gillette.

Groupoid Dynamical Systems

Definition

A groupoid dynamical system $(\mathscr{A}, G, \alpha)$ consists of an upper semicontinuous C^*-bundle $\mathscr{A}$ (such that $A = \Gamma_0(\mathscr{A})$ is separable), second countable locally compact groupoid G and a family $\{\alpha_\gamma\}_{\gamma \in G}$ such that

1. $\alpha_\gamma : A\big(s(\gamma)\big) \to A\big(r(\gamma)\big)$ is an isomorphism,
2. $\alpha_{\eta\gamma} = \alpha_\eta \circ \alpha_\gamma$ if $(\eta, \gamma) \in G^{(2)}$ and such that
3. $\gamma \cdot a := \alpha_\gamma(a)$ is a continuous G-action on $\mathscr{A}$.

Lemma

Suppose that $\{\alpha_\gamma\}_{\gamma \in G}$ satisfies (1) and (2). Then $(\mathscr{A}, G, \alpha)$ is a dynamical system if and only if $\alpha(f)(\gamma) := \alpha_\gamma\big(f(\gamma)\big)$ defines a $C_0(G)$-isomorphism of $s^*A \cong \Gamma_0(s^*\mathscr{A})$ onto $r^*A \cong \Gamma_0(r^*\mathscr{A})$.

Let $(\mathscr{A}, G, \alpha)$ be a groupoid dynamical system. The $\Gamma_c(r^*\mathscr{A})$ is a normed $*$-algebra:

$$f * g(\gamma) := \int_G f(\eta)\alpha_\eta\big(g(\eta^{-1}\gamma)\big)\, d\lambda^{r(\gamma)}(\eta)$$
$$f^*(\gamma) := \alpha_\gamma\big(f(\gamma^{-1})^*\big)$$
$$\|f\|_I := \max\Big\{\sup_u \int_G \|f(\gamma)\|\, d\lambda^u(\gamma), \sup_u \int_G \|f(\gamma^{-1})\|\, d\lambda^u(\gamma)\Big\}.$$

The universal C^*-norm is given by

$$\|f\| := \sup\{\, \|L(f)\| : L \text{ is a } \|\cdot\|_I\text{-decreasing representation}\,\}.$$

Then the associated crossed product is

$$\mathscr{A} \rtimes_\alpha G := \overline{\big(\Gamma_c(r^*\mathscr{A}), \|\cdot\|\big)}.$$

An Example

Example

Let H be a locally compact group and let X be a left H-space. Suppose that $A = \Gamma_0(\mathscr{A})$ is a $C_0(X)$-algebra and that $\sigma : H \to \operatorname{Aut} A$ is an ordinary (i.e., group) dynamical system such that $\sigma_h(f \cdot a) = \operatorname{lt}_h(f) \cdot \sigma_h(a)$, where $\operatorname{lt}_h(f)(x) = f(h^{-1} \cdot x)$. Let $\mathscr{G} = H \times X$ be the associated transformation group groupoid. Then we get a groupoid dynamical system $(\mathscr{A}, \mathscr{G}, \alpha)$ where $\alpha_{(h,x)}\big(f(h^{-1} \cdot x)\big) = \sigma_h(f)(x)$ for $f \in A = \Gamma_0(\mathscr{A})$. Then

$$\mathscr{A} \rtimes_\alpha \mathscr{G} \cong A \rtimes_\sigma H.$$

Equivalence for Dynamical Systems

Definition

An equivalence between groupoid dynamical systems $(\mathscr{B}, H, \beta)$ and $(\mathscr{A}, G, \alpha)$ is a upper semicontinuous Banach bundle $p : \mathscr{E} \to X$ over a (H, G)-equivalence X together with $B\big(r(x)\big) – A\big(s(x)\big)$-imprimitivity bimodule structures on the fibres $\mathscr{E}_x = p^{-1}(x)$ and commuting H- and G-actions such that

1. (Continuity) The maps $(e, f) \mapsto {}_{\mathscr{B}}\langle e\,,\, f\rangle$ is continuous from $\mathscr{E} * \mathscr{E}$ to $\mathscr{B}$, and $(b, e) \mapsto b \cdot e$ is continuous from $\mathscr{B} * \mathscr{E}$ to $\mathscr{E}$. Similarly, ...
2. (Equivariance) $p(\eta \cdot e) = \eta \cdot p(e)$ and $p(e \cdot \gamma) = p(e) \cdot \gamma$.
3. (Compatibility) ${}_{\mathscr{B}}\langle \eta \cdot e\,,\, \eta \cdot f\rangle = \beta_\eta\big({}_{\mathscr{B}}\langle e\,,\, f\rangle\big)$ and $\eta \cdot (b \cdot e) = \beta_\eta(b) \cdot (\eta \cdot e)$. Similarly, ...
4. (Invariance) ${}_{\mathscr{B}}\langle e \cdot \gamma\,,\, f \cdot \gamma\rangle = {}_{\mathscr{B}}\langle e\,,\, f\rangle$ and ....

A Reflexive Example

Example

Any dynamical system $(\mathscr{A}, G, \alpha)$ is equivalent to itself via

$$p : r^*\mathscr{A} \to G,$$

where G is viewed as a (G, G)-equivalence. Note that

$$r^*\mathscr{A} = \{ (\gamma, a) \in G \times \mathscr{A} : r(\gamma) = p(a) \}.$$

Then we simply define actions and inner products as follows:

$$\begin{aligned} {}_{A(r(\gamma))}\big\langle (\gamma , a), (\gamma, b) \big\rangle &= ab^* & a \cdot (\gamma, b) &= (\gamma, ab) \\ \big\langle (\gamma , a), (\gamma, b) \big\rangle_{A(s(\gamma))} &= \alpha_\gamma^{-1}(a^* b) & (\gamma, b) \cdot a &= (\gamma, b\alpha_\gamma(a)) \\ \eta \cdot (\gamma, a) &= (\eta\gamma, \alpha_\eta(a)) & (\gamma, a) \cdot \eta &= (\gamma\eta, a). \end{aligned}$$

Theorem (Equivalence Theorem for Dynamical Systems)

Suppose that $p : \mathscr{E} \to X$ is an equivalence between the dynamical systems $(\mathscr{B}, H, \beta)$ and $(\mathscr{A}, G, \alpha)$. Then we can build a $\mathscr{B} \rtimes_\beta H - \mathscr{A} \rtimes_\alpha G$-imprimitivity bimodule out of $\Gamma_c(\mathscr{E})$. In particular, $\mathscr{B} \rtimes_\beta H$ and $\mathscr{A} \rtimes_\alpha G$ are Morita equivalent.

- There isn't time to say much about the proof. The key step is a deep "disintegration" result due to Renault.
- We'll settle for some examples that play a role in our Brauer semigroup project.

Examples: Morita Equivalence of Dynamical Systems

Definition

We say that $(\mathscr{A}, G, \alpha)$ and $(\mathscr{B}, G, \beta)$ are Morita equivalent if there is a $\Gamma_0(\mathscr{A})-\Gamma_0(\mathscr{B})$-imprimitivity bimodule of the form $\mathsf{X} = \Gamma_0(\mathscr{X})$, for an upper semicontinuous bundle $q : \mathscr{X} \to G^{(0)}$ such each fibre $\mathsf{X}_u = q^{-1}(u)$ is an $A(u)-B(u)$-imprimitivity bimodule, and a G-action $\gamma \cdot x := V_\gamma(x)$ on X such that

$$ {}_{\mathscr{A}}\big\langle V_\gamma(x)\,,\, V_\gamma(y)\big\rangle = \alpha_\gamma\big({}_{\mathscr{A}}\langle x\,,\, y\rangle\big) \quad \text{and} $$
$$ \big\langle V_\gamma(x)\,,\, V_\gamma(y)\big\rangle_{\mathscr{B}} = \beta_\gamma\big(\langle x\,,\, y\rangle_{\mathscr{B}}\big) $$

These systems are equivalent via $\mathscr{E} := r^*\mathscr{X} = \{\,(\gamma, x) : r(\gamma) = q(x)\,\}$, where

$$ {}_{\mathscr{A}}\big\langle(\gamma\,,\, x), (\gamma, y)\big\rangle = {}_{\mathscr{A}}\langle x\,,\, y\rangle \qquad \big\langle(\gamma\,,\, x), (\gamma, y)\big\rangle_{\mathscr{B}} = \beta_\gamma^{-1}\big(\langle x\,,\, y\rangle_{\mathscr{B}}\big) $$
$$ a \cdot (\gamma, x) = (\gamma, a \cdot x) \qquad (\gamma, x) \cdot b = (\gamma, x \cdot \beta_\gamma(b)) $$
$$ \sigma \cdot (\gamma, x) = (\sigma\gamma, V_\sigma(x)) \qquad (\gamma, x) \cdot \sigma = (\gamma\sigma, x). $$

- We let $S(G)$ be the collection of Morita equivalence classes of dynamical systems $(\mathscr{A}, G, \alpha)$.
- In fact, $S(G)$ is a semigroup with respect to a product based on the balanced tensor product. ($S(G)$ is called the Brauer semigroup of G.)
- If X is a (H, G)-equivalence, then there is a semigroup isomorphism $\phi^X : S(G) \to S(H)$ defined by $\phi^X(\mathscr{A}, G, \alpha) = (\mathscr{A}^X, H, \alpha^X)$,
- where $\mathscr{A}^X$ is the quotient of $s_X^*\mathscr{A}$ be the G-action $(x, a) \cdot \gamma := (x \cdot \gamma, \alpha_\gamma^{-1}(a))$, and where
- $\alpha_h^X([x, a]) = [h \cdot x, a]$.

Examples: The Fundamental Construction

Example

To see that $(\mathscr{A}, G, \alpha)$ and $\phi^X(\mathscr{A}, G, \alpha) = (A^X, H, \alpha^X)$ are equivalent, let

$$\mathscr{E} := s_X^*\mathscr{A} = \{\, (x, a) : s(x) = p(a) \,\}.$$

Then we define

$$\begin{aligned} {}_{\mathscr{A}^X}\big\langle (x\ ,\ a), (x, b) \big\rangle &= [x, ab^*] & \big\langle (x\ ,\ a), (x, b) \big\rangle_{\mathscr{A}} &= a^* b \\ [x \cdot \gamma, b] \cdot (x, a) &= (x, \alpha_\gamma(b)a) & (x, a) \cdot b &= (x, ab) \\ h \cdot (x, a) &= (h \cdot x, a) & (x, a) \cdot \gamma &= (x \cdot \gamma, \alpha_\gamma^{-1}(a)). \end{aligned}$$

Thus $\mathscr{A} \rtimes_\alpha G$ is Morita equivalent to $\mathscr{A}^X \rtimes_{\alpha^X} H$.

Summary and Current Problems

1. I hope that I have convinced you that the equivalence theorem for dynamical systems is a powerful tool.
2. In addition to the examples mentioned here, it generalizes Raeburn's Symmetric Imprimitivity Theorem for ordinary crossed products.
3. The examples illustrated here will form an important part of our results on the Brauer Semigroup mentioned earlier. (Joint work with Jon Brown and Paul Muhly.)
4. These ideas admit an interesting generalization to Fell bundles which are a generalization of Fell's $*$-algebraic bundles over groups treated in Fell & Doran's two volume work. (This is the "beyond" mentioned in the title. These results appear in a preprint with Paul Muhly.)

References

[1] Paul S. Muhly and Dana P. Williams, *Equivalence and disintegration theorems for fell bundles and their C^*-algebras*, pre-print (2008). arXiv: 0806.1022.

[2] ______, *Renault's equivalence theorem for groupoid crossed products*, NYJM Monographs, vol. 3, State University of New York University at Albany, Albany, NY, 2008. Available at http://nyjm.albany.edu:8000/m/2008/3.htm.