p-Operator Spaces

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Operator Spaces

Operator spaces are spaces of operators on Hilbert spaces.

A concrete operator space is a norm closed subspace of $B(H)$ together with a mantrix norm $\| \cdot \|_n$ on each $M_n(V)$ given by

$$M_n(V) \subseteq M_n(B(H)) \cong B(\ell_2^n(H)).$$

Theorem [R 1988]: Let $V$ be a Banach space with a norm $\| \cdot \|_n$ on each matrix space $M_n(V)$. Then $V$ is completely isometrically isomorphic to a concrete operator space if and only it satisfies

M1. $\left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}$

M2. $\|\alpha x \beta\|_n \leq \|\alpha\| \|x\|_n \|\beta\|$

for all $x \in M_n(V), y \in M_m(V)$ and $\alpha, \beta \in M_n(C') = B(\ell_2^n)$. 
**Completely Bounded Maps**

Let $\varphi : V \rightarrow W$ be a bounded linear map. For each $n \in \mathbb{N}$, we can define a linear map

$$\varphi_n : [x_{ij}] \in M_n(V) \rightarrow [\varphi(x_{ij})] \in M_n(W).$$

The map $\varphi$ is called *completely bounded* if

$$\|\varphi\|_{cb} = \sup\{\|\varphi_n\| : n \in \mathbb{N}\} < \infty.$$

We let $CB(V,W)$ denote the space of all completely bounded maps from $V$ into $W$, which is again an operator space with matrix norm given by

$$M_n(CB(V,W)) = CB(V,M_n(W)).$$

In particular, the dual space $V^* = CB(V,\mathbb{C})$ has a natural operator space matrix norm given by

$$M_n(V^*) = CB(V,M_n(\mathbb{C})).$$
In the category of operator spaces, it is important to consider

**Matrix Norms** and **Completely Bounded Maps**
Examples of Operator Spaces:

- $C^*$-algebras or von Neumann algebras are operator spaces.

In particular, if $G$ is a locally compact/discrete group,

$$C^*_\lambda(G'), C^*(G') \text{ and } VN(G)$$

- Duals $A^*$ of $C^*$-algebras and preduals $M_*$ of von Neumann algebras

$$A(G) = VN(G)^*, B_\lambda(G) = C^*_\lambda(G)^*, \text{ and } B(G) = C^*(G)^*$$

- Moreover, Herz-Schur multiplier algebra $M_{cb}A(G) \subseteq CB(A(G), A(G))$:

$$M_{cb}A(G) = \{ \varphi : G \to C : m_\varphi : \psi \in A(G) \to \varphi\psi \in A(G) \text{ with } \|m_\varphi\|_{cb} < \infty \}.$$
Approximation Properties

A C*-algebra $A$ is said to be **nuclear** if there exists completely contractive maps $\varphi_\alpha : A \to M_{n(\alpha)}$ and $\psi_\alpha : M_{n(\alpha)} \to A$ such that

$$\psi_\alpha \circ \varphi_\alpha \to id_A$$

in the point-norm topology.

A C*-algebra $A$ is said to have the **CBAP** (resp. **CCAP**) if there exists finite rank maps $\varphi_\alpha : A \to A$ such that $\|\varphi_\alpha\|_{cb} \leq k$ (resp. $\|\varphi_\alpha\|_{cb} \leq 1$) and $\varphi_\alpha \to id_A$ in the point-norm topology.

A C*-algebra is said to have the **OAP** if for every $x = [x_{ij}] \in K(\ell_2) \hat{\otimes} A$ and $\varepsilon > 0$, there exists a finite rank map $T : A \to A$ such that

$$\|[T(x_{ij})] - [x_{ij}]\|_{K(\ell_2) \hat{\otimes} A} < \varepsilon.$$ 

A C*-algebra is said to be **exact** if we have the short exact sequence

$$0 \to K(\ell_2) \hat{\otimes} A \to B(\ell_2) \hat{\otimes} A \to Q(\ell_2) \hat{\otimes} A \to 0,$$

where $Q(H) = B(\ell_2)/K(\ell_2)$.  

Let $G$ be a discrete group.

We have the following implications for group $C^*$-algebra $C^*_\lambda(G)$

Nuclearity $\Rightarrow$ CCAP/CBAP $\Rightarrow$ OAP $\Rightarrow$ Exactness.

If we consider the Fourier algebra $A(G)$, then we have

- $C^*_\lambda(G)$ is nuclear $\iff G$ is amenable $\iff A(G)$ has a BAI/CAI.
- $C^*_\lambda(G)$ has the CBAP/CCAP $\iff A(G)$ has the CBAP/CCAP
- $C^*_\lambda(G)$ has the OAP $\iff A(G)$ has the OAP
- $C^*_\lambda(G)$ is exact $\iff A(G)$ has ???.
Finite Representatibility in $\{M_n\}$ and in $\{T_n\}$

**Theorem [Pisier 1995]:** A C*-algebra $A$ is exact if and only if it is finitely representable in $\{M_n\}$, i.e. for every f.d. subspace $E \subseteq A$ and $\varepsilon > 0$, there exist $n(\varepsilon) \in \mathbb{N}$ and $F \subseteq M_{n(\varepsilon)}$ such that

$$E \overset{1+\varepsilon}{\cong}_{cb} F,$$

i.e., there exists a linear isomorphism $T : E \to F$ such that

$$\|T\|_{cb} \|T^{-1}\|_{cb} < 1 + \varepsilon.$$

**Question:** It is natural to ask

- whether a C*-algebra $A$ is exact if and only if $A^*$ is finitely representable in $\{T_n\}$;

- whether $C^*_\lambda(G)$ is exact if and only if $A(G)$ is finitely representable in $\{T_n\}$. 
Actually, it is shown

**Theorem [Effros- Junge-Ruan 2000]:** $A^*$ (resp., $M_*$) is finitely representable in $\{T_n\}$ if and only if $A$ (resp., $M$) has QWEP.

**[Theorem Kirchberg 1993]:** QWEP is equivalent to Connes embedding conjecture.

**A. Connes’ conjecture 1976:** Every finite von Neumann algebra with separable predual is $*$-isomorphic to a von Neumann subalgebra of the ultrapower of the hyperfinite $II_1$ factor.
Operator Algebras on $L_p$ Spaces

Let $G$ be a locally compact group with a left invariant Haar measure. For $1 \leq p \leq \infty$, we may consider

$$L_p(G) = \{ f : G \to C : \text{measurable } \| f \|_p < \infty \}.$$ 

For $p = 1$, we get the convolution algebra $L_1(G)$ and

for $p = \infty$ we get the commutative von Neumann algebra $L_\infty(G)$.

For general $1 < p < \infty$, we can consider a (regular) representation $\lambda^p : G \to B(L_p(G))$ defined by

$$\lambda^p_s(\xi)(t) = \xi(s^{-1}t)$$

for $\xi \in L_p(G)$. For $f \in L_1(G)$, we have

$$\lambda^p(f) = \int_G f(s) \lambda^p_s ds.$$
We let $PF_p(G) = \{ \lambda^p(f) : f \in L_1(G) \}^{-\| \cdot \|} \subseteq B(L_p(G))$ and

et $PM_p(G) = \text{span}\{ \lambda^p_s : s \in G \}^{-w.o.t} \subseteq B(L_p(G))$.

We note that $PM_p(G)$ is a dual space with a predual, the Figà-Talamanca-Herz algebra,

$A_p(G) = \{ f : G \to C : f(s) = \sum_n \langle \eta_n, \lambda^p_s(\xi_n) \rangle \text{ with } \sum \| \eta_n \|_{L_p'} \| \xi_n \|_{L_p} < \infty \}$.

If we let

$\Lambda_p : \eta \otimes \xi \in L_p'(G') \otimes^\pi L_p(G') \to \langle \eta, \lambda^p_s(\xi) \rangle$.

then we can identify

$A_p(G) = L_p'(G') \otimes^\pi L_p(G')/\ker \Lambda_p$

with the quotient of $L_p'(G') \otimes^\pi L_p(G)$.

It is known (by Herz 1971) that $A_p(G)$ is a commutative Banach algebra and $G$ is amenable if and only if $A_p(G)$ has a BAI.
### Classical Case

- $C_0(G) \subseteq L_\infty(G)$

### $p=2$ Case

- $C^*_\lambda(G) \subseteq VN(G)$

### $1 < p < \infty$ Case

- $PF_p(G) \subseteq PM_p(G)$

### Other Inclusions

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<th>Classical Case</th>
<th>$p=2$ Case</th>
<th>$1 &lt; p &lt; \infty$ Case</th>
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<tr>
<td>$L_1(G)$</td>
<td>$M(G)$</td>
<td>$A(G) \subseteq B_\lambda(G) \subseteq B(G)$</td>
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<td>$\subseteq M_{cb}A(G) \subseteq MA(G)$</td>
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where $M(G)$ is the **measure algebra** on $G$,

$MA_p(G)$ is the **multiplier algebra** of $A_p(G)$,

and $M_{cb}A_p(G)$ is the **completely bounded multiplier algebra** on $A_p(G)$.

For $p = 2$, $M_{cb}A_2(G) = M_{cb}A(G)$ is the completely bounded **Herz-Schur multiplier algebra** on $G$.

What can we say about the corresponding properties on $p$-cases?
Uniformly Embeddability

Let \((\Omega, d)\) be a metric space. Then \((\Omega, d)\) is said to be uniformly embeddable into a Banach space \(V\) if there exist a map \(f : \Omega \to V\) and two monotone increasing functions \(\rho_1 \leq \rho_1\) on \([0, \infty) \to [0, \infty)\) such that

1) \(\rho_1(d(x, y)) \leq \|f(x) - f(y)\| \leq \rho_2(d(x, y))\)

2) \(\lim_{t \to \infty} \rho_1(t) = \infty\).

It is clear that

- Uniformly embeddable into Hilbert spaces
  \(\Rightarrow\) Uniformly embeddable into \(L_p\) spaces
  \(\Rightarrow\) Uniformly embeddable into uniformly convex Banach spaces.

**Theorem [Nowak 2005]** Let \((X, d)\) be a metric space. Then \((X, d)\) is uniformly embeddable into \(L_p\) space for some \(1 \leq p < 2\) if and only if \((X, d)\) is uniformly embeddable into \(H\).
Remark: For $p > 2$, this is false for general metric space: Johnson and Randrianarivony 2006 showed that $\ell_p$ does not uniformly embeddable into $H$.

However, it is still an open question for finitely generated groups $(G, d_l)$. 
Let $G$ be a finitely generated group with a finite generator set $S = S^{-1}$. Then we can obtain a length function

$$l(x) = \min \{ n : x = g_1 \cdots g_n, g_i \in S \}$$

and a metric $d_l(x, y) = l(x^{-1}y)$ on $G$. Then we can consider uniformly embedding problem for $(G, d_l)$. 
p-Operator Spaces

What are p-operator spaces?

One possible definition is the

“spaces of operators on $L_p$-spaces”

i.e.

“subspaces of $B(L_p(\mu))$”.

These are spaces we are interested in this talk!
General Definition of $p$-Operator Spaces

Let $SQ_p$ denote the set of all subspaces of quotients of $L_p$-spaces, which is equal to the set $QS_p$ of all quotients of subspaces of $L_p$-spaces.

Let $E \in SQ_p$ and $n \in N$, we define $p$-column norm on $E^n = \ell_p^n(E)$, i.e. we define

$$\| [x_i] \|_p = \left( \sum \| x_i \|^p \right)^{\frac{1}{p}}.$$

We define $M_n(B(E)) = B(E^n)$.

A concrete $p$-operator space $V$ is a norm closed subspace of $B(E)$ for some $E \in SQ_p$ together with a mantrix norm $\| \cdot \|_n$ on each $M_n(V)$ given by

$$M_n(V) \subseteq M_n(B(E)) \cong B(\ell_p^n(E)).$$
Theorem [Le Merdy 1996]: Let $V$ be a Banach space with a norm $\| \cdot \|_n$ on each matrix space $M_n(V)$. Then $V$ is completely isometrically isometric to a $p$-concrete operator space if and only it satisfies

M1. $\left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}$

Mp2. $\|\alpha x \beta\|_n \leq \|\alpha\| \|x\|_n \|\beta\|$ 

for all $x \in M_n(V), y \in M_m(V)$ and $\alpha, \beta \in M_n(C) = B(\ell_p^n)$. 

**p-Completely Bounded Maps**

Let $V$ and $W$ be $p$-operator spaces and let $\varphi : V \to W$ be a bounded linear map. Then $\varphi$ is called **$p$-completely bounded** if

$$\|\varphi\|_{pcb} = \sup\{\|\varphi_n\| : n \in \mathbb{N}\} < \infty.$$  

We let $CB_p(V, W)$ denote the space of all $p$-completely bounded maps from $V$ into $W$, which is again a $p$-operator space with matrix norm given by

$$M_n(CB_p(V, W)) = CB_p(V, M_n(W)).$$

In particular, the dual space $V^* = CB_p(V, \mathbb{C})$ has a natural $p$-operator space matrix norm given by

$$M_n(V^*) = CB_p(V, B(\ell_p^n)).$$

Now we can consider the $p$-analogues of operator spaces.
Second Duals

Let $V$ be a p-operator space. We can consider the natural dual matricial norm induced on $V^{**}$, i.e. we have

$$M_n(V^{**}) = CB(V^*, B(\ell_p^n)).$$

It is clear that the canonical inclusion

$$\kappa : x \in V \rightarrow \hat{x} \in V^{**}$$

is an isometry and is a complete contraction.

**Theorem [Daws 2007]** For a p-operator space $V$, the canonical inclusion

$$x \in V \rightarrow \hat{x} \in V^{**}$$

is a p-completely isometric injection i.e. for every $[x_{ij}] \in M_n(V)$, we have

$$\|[x_{ij}]\|_{M_n(V)} = \sup\{\|\Phi(x_{ij})\| : \text{all c.c. } \Phi = [\Phi_{kl}] \in M_m(V^*)_1, m \in \mathbb{N}\}$$

if and only if $V \subseteq B(L_p)$ for some $L_p$-space.
Daws showed that

- Every dual $p$-operator space $V^*$ is a $p$-operator subspace of $B(L_p)$.

- There exists a finite dimensional $p$-operator space $V$ which is not a $p$-operator subspace of any $B(L_p)$, i.e. the $p$-matrix norm on $V$ is different from the induced dual matrix norm on $V^{**} = V$ !

- Another problem is that we do not have the Arveson-Wittstock-Hahn-Banach extension theorem for general $p$-operator spaces., or for $p$-operator spaces $V \subseteq B(L_p)$.
Extension Theorem

Hahn-Banach Extension Theorem:

\[
\begin{array}{c}
W \\
\uparrow \\
V \\
\varphi \\
\downarrow \\
\tilde{\varphi} \\
\end{array} 
\quad \quad \quad 
\begin{array}{c}
\longrightarrow \\
C \\
\end{array}
\]

with \( \|\tilde{\varphi}\| = \|\varphi\| \).

Arveson-Wittstock-Hahn-Banach Extension Theorem:

\[
\begin{array}{c}
W \\
\uparrow \\
V \\
\varphi \\
\downarrow \\
\tilde{\varphi} \\
\end{array} 
\quad \quad \quad 
\begin{array}{c}
\longrightarrow \\
B(H) \\
\end{array}
\]

with \( \|\tilde{\varphi}\|_{cb} = \|\varphi\|_{cb} \).
Tensor Products for p-operator Spaces

Le Merdy 1996:  Haagerup tensor product $\otimes^{hp}$ is projective, but not necessarily injective.

Daws 2007:  Projective tensor product $\otimes^{\wedge p}$ is projective and we have the p-complete isometry

$$CB_p(V \hat{\otimes}^p W, Z) = CB_p(V, CB_p(W, Z)).$$

In particular,

$$(V \hat{\otimes}^p W)^* = CB_p(V, W^*).$$
Injective Tensor Product

We can define the injective tensor product

\[ V \hat{\otimes}^p W \subseteq CB(V^*, W). \]

If \( W \subseteq B(L_p) \), then for \([u_{ij}] \in M_n(V \otimes W)\), we can write

\[ \|[u_{ij}]\|_{\vee p} = \sup\{\|[\Phi \otimes \Psi](u_{ij})]\| : \Phi \in M_k(V^*)_1, \Psi \in M_l(W^*)\}. \]

**Theorem [An and R 2008]:** Let \( V \) and \( W \) be subspaces of \( B(L_p) \) spaces. Then \( V \hat{\otimes}^p W \) is again a subspace of some \( B(L_p) \).

**Remark:** However, it is still not clear whether \( \otimes^\vee p \) is injective, due to the lack of \( p \)-analogue of the Hahn-Banach extension theorem.
Let $E \in SQ_p$. We may define the column (respectively, row) $p$-operator space structure $E^c = B(C, E)$ (respectively, $E^r = B(E^*, C)$). More precisely, we have

$$M_n(E^c) = B(\ell^n_p, \ell^n_p(E))$$

(respectively, $M_n(E^r) = B(\ell^n_p(E^*), \ell^n_p)$).

**Theorem [An and R 2008]** Let $V \subseteq B(L_p)$. We have the $p$-complete isometries

- $V \otimes^{hp} L_p^c = V \otimes^{\wedge p} L_p^c$ and $L_p^r \otimes^{hp} V = L_p^r \otimes^{\wedge p} V$

- $L_p^c \otimes^{hp} V = L_p^c \otimes^{\vee p} V$ and $V \otimes^{hp} L_p^r = V \otimes^{\vee p} L_p^r$
Combining the above result, we get the following $p$-complete isometric isomorphisms

- $T(L_p) \otimes \wedge^p V = L_p^r \otimes \wedge^p V \otimes \wedge^p L_p^c = L_p^r \otimes h_p V \otimes h_p L_p^c$
- $K(L_p) \otimes \vee^p V = L_p^c \otimes \vee^p V \otimes \vee^p L_p^r = L_p^c \otimes h_p V \otimes h_p L_p^r$.

In particular, we get

- $L_p^c \otimes h_p L_p^r = L_p^c \otimes \vee^p L_p^r = K(L_p)$ and $L_p^r \otimes h_p L_p^c = L_p^r \otimes \wedge^p L_p^c = T(L_p)$.
- $K(L_p)^* = T(L_p)$ and $T(L_p)^* = B(L_p)$.

**Remark:** Though these results are quite similar to operator spaces, the proofs have already need some quite different techniques. For instance, to show $K(L_p)^* = T(L_p)$ for separable $L_p$ space, we need the fact:

$L_p$ is separable dual space with RNP and approximation property.

Therefore, every integral map is nuclear, i.e. we have

$$K(L_p)^* = (L_p' \otimes^e L_p)^* = I(L_p, L_p) = N(L_p, L_p) = T(L_p).$$
It is known (from definition of $\hat{\otimes}^p$) that for each $n \in \mathbb{N}$

$$(T_n \hat{\otimes}^p V)^* = CB_p(V, B(\ell_p^n)) = M_n(V^*) = M_n \hat{\otimes}^p V^*.$$

However, it is quite interesting to show

**Theorem [An and R 2008]:** Let $V$ be a subspace of $B(L_p)$. For each $n \in \mathbb{N}$, we have the isometry

$$(M_n \hat{\otimes}^p V)^* = T_n \hat{\otimes}^p V^*.$$

Moreover, we have the complete isometry

$$(K(L_p) \hat{\otimes}^p V)^* = T(L_p) \hat{\otimes}^p V^*.$$
p-Operator Approximation Property

We can study the corresponding approximation properties for p-operator spaces.

For instance, let $V$ be a subspace of $B(L_p)$, then $V$ is said to have p-operator space approximation property (p-OAP) if for every $x = [x_{ij}] \in K(\ell_p) \otimes^p V$ and $\varepsilon > 0$, there exists a finite rank map $T : V \to V$ such that

$$
\| [T(x_{ij})] - [x_{ij}] \|_{K(\ell_p) \otimes^p V} < \varepsilon.
$$

**Theorem [An and R 2008]:** Let $V$ be a p-operator subspace of some $B(L_p)$. Then TFAE:

1) $V$ has the p-OAP;

2) the canonical map $\Phi : V \hat{\otimes}_p V^* \to V \otimes^p V^*$ is injective;

3) if $u \in V \hat{\otimes}_p V^*$ is such that $\Phi(u) = 0$, then $\text{trace}(u) = 0$. 
Now let $G$ a locally compact group. Then $M_{cb}A_p(G) = \{\varphi : G \to \mathbb{C}, m_\varphi : \psi \in A_p(G) \to \varphi\psi \in A_p(G')$ with $\|m_\varphi\|_{cb} < \infty\}$. 

**Theorem [Daws 2007]:** A continuous function $\varphi : G \to C$ is a cb multiplier of $A_p(G)$ if and only if there exist bounded maps $\xi : G \to E$ and $\eta : G \to E'$ (with $E$ in $SQ_p$ and $E'$ in $SQ_{p'}$) such that

$$\varphi(st^{-1}) = < \eta(s), \xi(t) >.$$

Moreover we have $\|m_\varphi\|_{pcb} = \inf\{\|\eta\|_\infty \|\xi\|_\infty\}$. 

**Proposition [Miao and R 08]:** $M_{cb}A_p(G)$ is a dual space with a predual $Q_p(G)$. 

Therefore, we can define $p$-AP for $G$: 

1 is contained in the $\sigma(M_{cb}A_p(G), Q_p(G))$ closure of $A_p(G)$ in $M_{cb}A_p(G)$. 

Now we are working on the $p$-OAP for $FP_p(G)$ and $A_p(G)$. Most of results by Haagerup-Kraus 1994 can be generalized to $p$-case.
Thank you for your attention!