Operator space structure of $JC^*$-triples

Richard M. Timoney
Trinity College Dublin

Les Bunce (Reading) & Brian Feely (TCD)

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\( JC^* \)-triples and TROs

• on \( \mathcal{B}(H) \), we define the \textit{Jordan triple product}
\[ \{x_1, x_2, x_3\} = (x_1 x_2^* x_3 + x_3 x_2^* x_1)/2 \]
A \( JC^* \)-triple is \( X \subset \mathcal{B}(H) \) (closed) subspace, closed under \( \{\cdot, \cdot, \cdot\} \).

• \textit{Ternary product} on \( \mathcal{B}(H) \) is \[ [x_1, x_2, x_3] = x_1 x_2^* x_3 \]
A \textit{TRO} is \( T \subset \mathcal{B}(H) \) (closed) subspace, closed under \([\cdot, \cdot, \cdot]\).
Examples

(i) $A \subset B(H)$ a $C^*$-algebra $\Rightarrow A$ a TRO $\Rightarrow A$ a $JC^*$-triple

(ii) $T$ a finite dimensional TRO $\Rightarrow T$ a direct sum of rectangular matrix spaces

(e.g. $M_{5,7}(\mathbb{C})$, $M_{8,1}(\mathbb{C})$).

(iii) $X = \{x \in M_n(\mathbb{C}) : x^t = x\}$ is $JC^*$-triple (not TRO).

Note: $X =$ symmetric $n \times n$ matrices.
Background motivation — $JC^*$-triples

Consider $JC^*$-(sub)triples $X \subset \mathcal{B}(H)$ with operator norm. $B_X = \{x \in X : \|x\| < 1\}$ unit ball

$\text{Aut}(B_X)$ transitive (contains a Möbius like group).

$X$ Banach (over $\mathbb{C}$), $\text{Aut}(B_X)$ transitive $\iff X$ has a $JB^*$-triple product. (W. Kaup 1983). $B_X$ a bounded symmetric domain. (Original Lie-theoretic approach by Cartan (1932) for $\dim X < \infty$. Jordan approach due to M. Koecher (1969).)

‘Most’ $JB^*$-triples are (isometric to) $JC^*$-triples.
Notions of equivalence?

Biholomorphic equivalence of bounded symmetric domains

Linear isometry of (unit balls of) $JB^*$-triples

Algebraic ($JB^*$-triple) isomorphism

Conclusion: We treat two $JC^*$-triples as the ‘same’ if they are (linearly) isometric
Background motivation — TROs

Associative product \[ [x_1, x_2, x_3] = x_1 x_2^* x_3 \]

Characterised by Zettl (1983) in abstract algebraic terms. Left \( C^* \)-algebra of \( T \) is \( \mathcal{L}_T \) (generated by \( x_1 x_2^* \)). Right \( C^* \)-algebra of \( T \) is \( \mathcal{R}_T \) (generated by \( x_1^* x_2 \)). Algebraic TRO morphisms extend to left and right \( C^* \)-algebras.

Isometry of TROs does not imply algebraic TRO isomorphism. Simplest examples: \( M_{2,1}(\mathbb{C}) \) and \( M_{1,2}(\mathbb{C}) \).

Language of operator spaces gives another viewpoint.
TROs in operator space theory

- TROs $T_1$ and $T_2$ are the same $\iff$ completely isometric.

- TRO structure $\Rightarrow$ unique compatible operator space structure

- (Hamana theory) every operator space can be canonically embedded in its injective envelope and that envelope is a TRO.
Is a $JC^*$-triple an operator space?

Answer: Not so obviously (or canonically) 
Too obvious: $X \subset \mathcal{B}(H)$ closed under triple product $\Rightarrow X$ an operator space.

The snag is that two $JC^*$-triples are considered the ‘same’ if they are isometric and this does not imply complete isometry. [Neal & Russo]

e.g. the TROs $M_{2,1}(\mathbb{C})$ and $M_{1,2}(\mathbb{C})$ are the same as $JC^*$-triples, but not as operator spaces. Isometric but not completely isometric.
Universal TRO

**Theorem:** Given a $JC^*$-triple $X$, there is a unique universal embedding $\alpha_X : X \to T^*(X)$ into TRO s.t.

- $\alpha_X$ is a $JC^*$-triple isom onto its range
- $T^*(X) = \text{(closure of)} \ subTRO \ generated \ by \ \alpha_X(X)$
- $\forall \ \pi : X \to T \ JC^*$-triple homom into TRO $T$, $\exists! TRO$-hom $\tilde{\pi} : T^*(X) \to T$ lifting $\pi$
  $\left( \tilde{\pi}(\alpha_X(x)) = \pi(x) \forall x \in X \right)$
\[ T^*(X) \]

\[ \alpha_X \uparrow \xrightarrow{\tilde{\pi}} \]

\[ X \xrightarrow{\pi} T \]

Consequence: Given a JC*-triple \( X \) (up to equivalence), for any realisation \( X \subset B(H) \) as a JC*-subtriple, \( \text{TRO}(X) \cong T^*(X)/\mathcal{I} \) for \( \mathcal{I} \subset T^*(X) \) a TRO ideal with \( \mathcal{I} \cap \alpha_X(X) = \{0\} \).

Each such \( \mathcal{I} \) yields an operator space structure \( X_{\mathcal{I}} \) on \( X \) — all possible JC*-operator space structures.
How to find $T^*(X)$?

Tripotents are a key concept in $JC^*$-triples $X: e \in X$ with $\{e, e, e\} = e$. Associated Peirce spaces $X_\lambda(e) = \{x \in X : 2\{e, e, x\} = \lambda x\}$ $X_\lambda(e)$ nonzero implies $\lambda \in \{0, 1, 2\}$.

$X_2(e)$ is a Jordan $*$-algebra: Jordan product $x \cdot y = \{x, e, y\}$, conjugation $x \mapsto \{e, x, e\}$

A tripotent $e \in X$ is called unitary if $X_2(e) = X$.

Propostion: If $\exists$ unitary tripotent $e \in X$, $T^*(X)$ ‘equals’ the universal $C^*$-algebra generated by the $J\mathcal{C}^*$-algebra structure. (computed in 1970s – by
Alfsen, Stormer, Hanche-Olsen).

- $X = M_n, \quad T^*(M_n) = M_n \oplus M_n, \quad \alpha_X(x) = x \oplus x^t \quad (n \geq 2)$

- $X = \mathcal{B}(H), \quad T^*(X) = \mathcal{B}(H) \oplus \mathcal{B}(H), \quad \alpha_X(x) = x \oplus x^t$

  TRO ideals with $\mathcal{I} \cap \alpha_X(X) = \{0\}$ are $\mathcal{I} = \{0\}$, (always gives $\text{MAX}_{JC}(X)$), $\mathcal{I} = \{0\} \oplus \mathcal{B}(H)$, $\mathcal{I} = \mathcal{B}(H) \oplus \{0\}$, $\mathcal{I} = \{0\} \oplus \mathcal{K}(H)$, $\mathcal{I} = \mathcal{K}(H) \oplus \{0\}$, \ldots

- $X = \{x \in \mathcal{B}(H) : x^t = x\} \quad (\text{dim} \geq 2) \quad T^*(X) = \mathcal{B}(H)$
\( X = \{ x \in \mathcal{B}(H) : x^t = -x \} \) (6 \leq \dim H, \dim H \text{ even or } \infty) \ T^*(X) = \mathcal{B}(H), \ \alpha_X(x) = x.

\( X = V_k = \text{spin factor } (k \geq 3) \ (\dim V_k = k + 1) \)
\( T^*(V_{2n}) = M_{2n}, \ T^*(V_{2n+1}) = M_{2n} \oplus M_{2n}, \ T^*(V_k) = \text{CAR if } k \text{ infinite.} \)

For the other Cartan factors (building blocks for \( JC^*\)-triples)

\( X = M_{k,n} = \mathcal{B}(H, K), \ T^*(X) = \mathcal{B}(H, K) \oplus \mathcal{B}(K, H), \ \alpha_X(x) = x \oplus x^t \ (2 \leq n < k) \)
• $X = \{ x \in M_{2n+1} : x^t = -x \} \ (n \geq 2)$, $T^*(X) = M_{2n+1}$, $\alpha_X(x) = x$.

• $X = \ell^2_k = M_{k,1}$, $T^*(X) = \bigoplus_{j=1}^{k} B \left( \Lambda^j(X), \Lambda^{j-1}(X) \right)$ contained in Fock space on dimension $k$ ($\alpha_X(X) =$ span of $k$ annihilation operators for o.n basis of $\ell^2_k$, operators satisfying CAR)

• $X = H$ general Hilbert space, $T^*(X) = \text{TRO}$ generated by embedding in Fock space.
Further results

$T^*$ is an exact functor $JC^*$-triples $\rightarrow$ TROs

$\dim X < \infty$ ($X$ a $JC^*$-triple) $\Rightarrow$ $\dim T^*(X) < \infty$

**Corollary** (results of Neal & Russo) Characterisation of (finite dimensional) $JC^*$-subtriples $X \subset B(H)$ up to complete isometry.

Interesting examples (say for injective envelopes)?